

LOCAL MODEL FOR THE MODULI SPACE OF AFFINE VORTICES

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ABSTRACT. We show that the moduli space of regular affine vortices has the structure of a smooth manifold. The construction uses Ziltener's Fredholm theory results [25].

1. INTRODUCTION

A vortex is a pair consisting of a connection on a bundle and a section holomorphic with respect to that connection. The pair satisfies an equation involving the curvature of the connection and a non-linear term dependent on the section. Affine vortices are vortices whose domain is the affine complex line. These objects initially arose in the classical situation of Ginzburg-Landau theory of superconductivity ([15], [12]), but since then they have occurred in various areas of mathematics and physics. For example, the moduli space of affine vortices is related to Weil's scheme of torsion quotients ([1], [2]). Gukov and collaborators have shown connections to knot invariants [5] and to Chern-Simons theory [10]. Our study of the moduli space of affine vortices is motivated by its applications in symplectic geometry, especially its role in the quantum Kirwan map defined by Woodward [20] and Ziltener [25].

Let G be a compact Lie group and X a Hamiltonian symplectic G -manifold. An affine vortex is a pair (A, u) , where A is a connection on the trivial principal bundle $P := \mathbb{C} \times G$ and u is a section on the associated bundle $P(X) := P \times_G X$ that satisfies the holomorphicity and vortex equations:

$$\bar{\partial}_A u = 0, \quad F_A + \Phi(u) \, \text{dvol}_{\mathbb{C}} = 0.$$

Here F_A is the curvature of the connection A and $\Phi : X \rightarrow \mathfrak{g}^{\vee} \simeq \mathfrak{g}$ is the moment map of the G -action. The energy of a vortex is

$$E(A, u) := \frac{1}{2} \int_{\mathbb{C}} |F_A|^2 + |\text{d}_A u|^2 + |\Phi(u)|^2 \, \text{dvol}_{\mathbb{C}}.$$

A complete classification of affine vortices is known in the toric case. The first result in this direction, by Taubes ([15], [12]), provides a complete description in the case when the target is the complex line with the standard action of S^1 . Recently, this result was generalized to the case of the standard S^1 action on \mathbb{C}^n by the second named author [22]. A classification in the general toric case was provided by Gonzalez-Woodward [9]. The first named author, in joint work with Woodward [17], proved a Hitchin-Kobayashi correspondence for affine vortices, in the case when the target has the structure of an affine or projective variety. This correspondence is

used in [20] to construct the moduli space of affine vortices via the Behrend-Fantechi machinery.

In this paper, we produce a manifold structure on the space of affine vortices via Fredholm theory. At a finite energy vortex (A, u) , we define an augmented Cauchy-Riemann differential operator, aka the vortex differential operator, consisting of 3 terms - the derivatives of the holomorphicity and vortex equations, and a vortex Coulomb gauge operator. A vortex is *regular* if the vortex differential operator is surjective. The following is the main result of this paper.

Theorem 1.1. *The moduli space of regular affine vortices representing an equivariant homology class $\beta \in H_2^G(X)$ modulo gauge transformations has the structure of a smooth manifold of dimension $\dim(X) - 2\dim(G) + 2c_1^G(\beta)$. Convergence in the manifold topology coincides with uniform convergence on compact sets modulo gauge.*

The study of affine vortices in symplectic geometry is motivated by their role in the definition of the quantum Kirwan map, which relates the gauged Gromov-Witten invariants of a Hamiltonian symplectic manifold and the ordinary Gromov-Witten invariants of the symplectic quotient. This relation was uncovered by Gaio-Salamon [8] when they considered the large area adiabatic limit of vortices over Riemann surfaces. As a consequence, Salamon and Ziltener [23] suggested a definition of the quantum Kirwan map by a count of affine vortices. Ziltener carried out various steps towards defining this map, including the compactification of the moduli space of affine vortices, the optimal asymptotic behavior, and the linear Fredholm theory, published in [24] and [25]. Meanwhile, Woodward defined the quantum Kirwan map in an algebro-geometric setting [20] using the formal setup of CohFT algebras set up in an earlier work [13].

A local model for the moduli space is constructed as the zero locus of a section $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{E}$ of a Banach bundle over a Banach manifold. In [25], Ziltener showed the augmented linearized operator associated to an affine vortex gives a Fredholm operator $\mathcal{D} : \mathcal{X} \rightarrow \mathcal{Y}$, but he did not realize \mathcal{X} as the tangent space of any Banach manifold. In this paper, we explicitly define a local chart $\mathcal{B}_{(A,u)}$ of a Banach manifold near an affine vortex (A, u) , whose tangent space coincides with the space \mathcal{X} Ziltener used. Therefore, using Ziltener's Fredholm result and the index formula (which we reprove in this paper), a manifold structure of the moduli space is obtained. Another nontrivial ingredient of our proof is to show that the topology defined by local convergence modulo gauge transformations coincides with the topology induced from $\mathcal{B}_{(A,u)}$.

A symplectic geometric description of the moduli space of affine vortices has the advantage of being generalizable to the *open* case, i.e. to the case of vortices defined on the upper half of the complex plane with boundary in a Lagrangian submanifold. The generalization of the current result to the open case will be included in a revised version of this paper. Conjectured by Woodward in [21], counting vortices on the half-plane leads to an *open quantum Kirwan map*, which is supposed to intertwine the A_∞ structures on X and its symplectic quotient $X//G$. The definition of this map is the ongoing project of the second named author (see [18]).

An outline of the organization of the paper and a sketch of the proof of the main Theorem are provided at the end of Section 2.

2. PRELIMINARIES

In this section we define concepts related to vortices and weighted Sobolev spaces.

2.1. Vortices. We first introduce notation for Hamiltonian group action on symplectic manifolds. Let G be a compact connected Lie group. Let (X, ω, J) be a symplectic manifold with a compatible almost complex structure J . Suppose G acts on X preserving the symplectic and almost complex structures (ω, J) . Suppose the action is *Hamiltonian*, namely, there is a *moment map*, i.e., a G -equivariant map $\Phi : X \rightarrow \mathfrak{g}^\vee$ such that

$$(1) \quad \omega(s_X, \cdot) = d\langle \Phi, s \rangle, \quad \forall s \in \mathfrak{g},$$

where $s_X(x) = \frac{d}{dt} \exp(t\xi)x|_{t=0} \in \text{Vect}(X)$ is given by the infinitesimal action of ξ on X . Since G is compact, \mathfrak{g} has an Ad-invariant metric. We fix such a metric and identify $\mathfrak{g} \simeq \mathfrak{g}^\vee$, so the moment map becomes a map $\Phi : X \rightarrow \mathfrak{g}$.

Assumption 2.1. *The G -action on $\Phi^{-1}(0)$ is free.*

The infinitesimal action of \mathfrak{g} on X , given by $\mathfrak{g} \ni s \mapsto s_X \in \text{Vect}(X)$, extends to a linear map $\mathfrak{g}_\mathbb{C} \rightarrow \text{Vect}(X)$ as $\mathfrak{g} \oplus i\mathfrak{g} \ni s_1 + is_2 \mapsto (s_1)_X + J(s_2)_X$.

We next describe connections, curvature, and their behavior under gauge transformations. Suppose Σ be a Riemann surface with metric. Let $P \rightarrow \Sigma$ be a principal G -bundle. A *connection* is a G -equivariant one-form $A \in \Omega^1(P, \mathfrak{g})^G$, that satisfies $A(\xi_P) = \xi$ for $\xi \in \mathfrak{g}$. The *space of connections* $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(\Sigma, P(\mathfrak{g}))$, where $P(\mathfrak{g}) = P \times_G \mathfrak{g}$ is the *adjoint bundle*. In case P is the trivial bundle $\Sigma \times G$, there is a trivial connection d , and the adjoint bundle has a trivialization $P(\mathfrak{g}) \simeq \Sigma \times \mathfrak{g}$. Then, the space of connections is

$$\mathcal{A}(P) = d + \Omega^1(\Sigma, \mathfrak{g}).$$

The *curvature* of a connection A is a two-form $F_A \in \Omega^2(\Sigma, P(\mathfrak{g}))$. In particular, on a trivial bundle, for a connection $A = d + a$,

$$F_A := da + [a \wedge a]/2 \in \Omega^2(\Sigma, \mathfrak{g}).$$

A *gauge transformation* is an automorphism of P that is an equivariant bundle map $P \rightarrow P$. On the trivial bundle $\Sigma \times G$, a gauge transformation is a map $g : \Sigma \rightarrow G$, and it acts on a connection $A = d + a$ as

$$g(A) = d + (gdg^{-1} + \text{Ad}_g a).$$

Differentiating, we see that the infinitesimal action of $\zeta : \Sigma \rightarrow \mathfrak{g}$ on A is $-d_A \zeta$.

Connections on principal bundles determine covariant derivatives on associated fiber bundles. On the associated bundle $P(X) = P \times_G X$, a connection $A \in \mathcal{A}(p)$ determines a splitting of the tangent space $TP(X) = T^{\text{vert}} P(X) \oplus \pi^* T\Sigma$, and hence a covariant derivative

$$d_A : \Gamma(P(X)) \rightarrow \bigcup_{u \in \Gamma(P(X))} \Omega^1(\Sigma, u^* T^{\text{vert}} P(X)).$$

For example on the bundle $\Sigma \times X$, writing $A = d + a$,

$$d_A u := du + a_u \in \Omega^1(\Sigma, u^*TX),$$

where at a point $z \in \Sigma$, $a_u(z)$ is the infinitesimal action of $a(z)$ at $u(z)$. The connection A also induces an almost complex structure on $P(X)$ as the direct sum $J_A := J_X \oplus j_\Sigma$. The $(0,1)$ -part of the covariant derivative d_A is the delbar operator $\bar{\partial}_A$. For a section $u \in \Gamma(\Sigma, P(X))$, the $(0,1)$ -derivative $\bar{\partial}_A u$ lies in the space $\Omega^{0,1}(\Sigma, u^*T^{vert}P(X))$.

A *gauged map* (A, u) from P to X consists of a connection A and a section u of $P(X)$. A *symplectic vortex* is a gauged map that satisfies

$$(2) \quad \bar{\partial}_A u = 0, \quad *F_A + \Phi(u) = 0.$$

Here $*$ is the Hodge star operator of the metric on Σ . In the case when Σ is \mathbb{C} , equipped with the standard Euclidean metric, a solution of (2) is called an *affine vortex*. The *energy* of a gauged map (A, u) is

$$E(A, u) := \frac{1}{2} \int_{\Sigma} (|F_A|^2 + |d_A u|^2 + |\Phi \circ u|^2) \, \text{dvol}_{\Sigma}.$$

The space of gauge equivalence classes of vortices over an open subset $B \subset \mathbb{C}$ is denoted by $M^G(B, X)$. On this space, we define a *compact convergence topology*. A sequence of affine vortices (A_i, u_i) converges to a limit (A_∞, u_∞) in the compact convergence topology if their images are contained in a compact subset of X , there exist gauge transformations $g_i : \mathbb{C} \rightarrow G$ such that $g_i(A_i, u_i)$ converges smoothly to (A_∞, u_∞) on compact subsets of \mathbb{C} , and the sequence of energies converge : $\lim_{i \rightarrow \infty} E(A_i, u_i) = E(A_\infty, u_\infty)$. We will later see that for $B = \mathbb{C}$, the energy convergence requirement can be replaced by the condition that the sequence of vortices belong to the same equivariant homology class for large i .

The moduli space $M^G(B, X)$ is realized as the zero set of a Fredholm section \mathcal{F} of a Banach bundle. The subset of the zero level set $\mathcal{F}^{-1}(0)$, where the derivative of \mathcal{F} is surjective, has the structure of a smooth manifold. The derivative of the holomorphicity and the vortex equations at a vortex (A, u) are given by

$$\begin{aligned} \mathcal{D}_{(A,u)} : \Omega^1(\mathbb{C}, \mathfrak{g}) \times \Gamma(\mathbb{C}, u^*TX) &\rightarrow \Omega^{0,1}(\mathbb{C}, u^*TX) \times \Gamma(\mathbb{C}, \mathfrak{g}), \\ \begin{pmatrix} \alpha \\ \xi \end{pmatrix} &\mapsto \begin{pmatrix} (\nabla_A^X \xi + \alpha_X)^{0,1} - \frac{1}{2} J(\nabla_\xi^X J) \partial_{J,A} u \\ d\Phi(u)\xi + *d_A \alpha \end{pmatrix}. \end{aligned}$$

In the definition of the above differential operators, the term $\nabla_A^X \xi$ involves differentiating sections of the bundle u^*TX over \mathbb{C} . This differentiation is carried out using the Levi-Civita connection ∇^X of the metric $\omega(\cdot, J\cdot)$ and the connection A on the principal bundle P . Assuming a local trivialization of P , where $A = d + a$, given a section $\xi \in \Gamma(\mathbb{C}, u^*TX)$, its derivative at $z \in \mathbb{C}$ along $v \in T_z \mathbb{C}$ is

$$\nabla_{A,v}^X \xi = \nabla_{du(v)} \xi + \nabla_\xi(a(v)_X).$$

Since the moduli space is defined by quotienting the space of vortices by gauge transformations, we include a gauge fixing condition. Given a vortex (A, u) , a gauged map $(A + \alpha, \exp_u \xi)$ is in *Coulomb gauge* with respect to (A, u) if $d_A^* \alpha + d\Phi(J\xi) = 0$. Here $d_A^* := *d_A^*$ is the formal adjoint of d_A and $(\alpha, \xi) \mapsto d_A^* \alpha + d\Phi(J\xi)$ is called the

vortex Coulomb gauge operator. A chart of the moduli space $M^G(\mathbb{C}, X)$ centered at (A, u) is constructed by restricting attention to gauged maps that are in Coulomb gauge with respect to (A, u) . The augmented differential operator is

$$(3) \quad \hat{D}_{(A,u)} : \Omega^1(\mathbb{C}, \mathfrak{g}) \times \Gamma(\mathbb{C}, u^*TX) \rightarrow \Omega^{0,1}(\mathbb{C}, u^*TX) \times \Gamma(\mathbb{C}, \mathfrak{g}) \times \Gamma(\mathbb{C}, \mathfrak{g}),$$

$$\begin{pmatrix} \alpha \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} (\nabla_A^X \xi + \alpha_X)^{0,1} - \frac{1}{2} J(\nabla_\xi J) \partial_{J,A} u \\ d\Phi(u)\xi + *d_A \alpha \\ d_A^* \alpha + d\Phi(J\xi) \end{pmatrix}.$$

2.2. Asymptotic behavior of vortices. Affine vortices have good asymptotic properties. The singularity at infinity can be removed in a certain weak sense, and u extends to a continuous section of an associated bundle $P(X)$ for some principal bundle $P \rightarrow \mathbb{P}^1$. It follows that we can define two maps. The first is a forgetful map hol , which maps a vortex to the equivariant homology class of the domain principal bundle $[P] \in H_2^G(\text{point})$. The second is an evaluation map at infinity ev_∞ . Both these maps are continuous under the compact convergence topology on the space of affine vortices. We start by stating an asymptotic decay result for energy density of affine vortices.

Proposition 2.2. (Energy density decay, [24, Corollary 1.4]) *Let (A, u) be a vortex on $\mathbb{C} \setminus B_R$ that has finite energy and bounded image. Then for any $\epsilon > 0$, there is a constant $c > 0$ such that*

$$(4) \quad |F_A(z)|^2 + |d_A u(z)|^2 + |\Phi(u(z))|^2 \leq c|z|^{-4+\epsilon} \quad \forall z \in \mathbb{C} \setminus B_R.$$

Further, if (A_i, u_i) is a sequence of vortices converging to (A_∞, u_∞) in the compact convergence topology, then the constants $c(i, \epsilon)$ can be chosen independent of i .

The last statement of the proposition is not part of the result in [24, Corollary 1.4], but the proof of (4) can be extended in a straightforward way to prove the existence of a uniform constant.

Definition 2.3. (Holonomy of a connection) Let A be a connection on the trivial G -bundle $\mathbb{C} \times G$. The holonomy of the connection on the circle of radius r , $\text{Hol}_A(r, \theta) \in G$ is obtained by parallel transporting the fiber at $(r, 0)$ to (r, θ) along the curve $\{|z| = r\}$ going in the counter-clockwise direction. If $A = d + a_r dr + a_\theta d\theta$, then $\text{Hol}_A(r, \theta)$ is given as the solution of the ODE

$$(5) \quad \text{Hol}_A(r, \theta)^{-1} \frac{\partial}{\partial \theta} \text{Hol}_A(r, \theta) = a_\theta(r, \theta), \quad \text{Hol}_A(r, 0) = \text{Id}.$$

Proposition 2.4 (Removal of singularity for vortices at infinity). *Let (A, u) be a vortex on $\mathbb{C} \setminus B_1$ that has finite energy and bounded image. Suppose A is in radial gauge, namely $A = d + a d\theta$ for $a : \mathbb{C} \setminus B_1 \rightarrow \mathfrak{g}$. Then there exist $x_0 \in \Phi^{-1}(0)$ and $k_0 \in W^{1,p}(S^1, G)$ satisfying the following condition. For any $p > 2$ and $0 < \gamma < \frac{2}{p}$, there exists a constant c such that*

$$\lim_{r \rightarrow \infty} \max_{\theta \in [0, 2\pi]} d(x_0, k_0(\theta))^{-1} u(re^{i\theta}) = 0,$$

$$\|a(r, \cdot) - k_0^{-1} \partial_\theta k_0\|_{L^p([0, 2\pi], G)} < cr^{-\gamma}.$$

Further, if (A_i, u_i) is a sequence of vortices converging to (A_∞, u_∞) in the compact convergence topology, then the constants $c(i, \gamma)$ can be chosen independent of i .

Proposition 2.4 is an improved version of [23, Proposition D.6]. The improved constants are obtained by applying the results in [24]. This modification is discussed in [17, Section 5].

Remark 2.5. (Equivariant homology class) Let (A, u) be an affine vortex. By Proposition 2.4, there is a G -bundle P on \mathbb{P}^1 such that u extends to a continuous section of $P(X)$. So u represents a class $[u] \in H_2^G(X)$. Using a calculation similar to [4, Theorem 3.1], we can derive

$$(6) \quad E(A, u) = \langle [\omega - \Phi], [u] \rangle,$$

where $[\omega - \Phi] \in H_G^2(X)$ is represented by the equivariant 2-form $\omega - \Phi$ and $\langle \cdot, \cdot \rangle$ is the pairing between equivariant homology and cohomology. Ziltener [23] proves that when a sequence of affine vortices converges to a (possibly reducible) limit, the equivariant homology class is preserved in the limit. Our main theorem yields a weaker result as a corollary. The existence of the moduli space of regular vortices implies that if a sequence of regular vortices converges to a regular vortex in the compact convergence topology, then the elements of the sequence and the limit have the same equivariant homology class, because they are part of a continuous family.

Remark 2.6. (Holonomy of a vortex) The projection map $(X \times EG)/G \rightarrow BG$ induces a map in equivariant homology $H_2^G(X) \rightarrow H_2^G(\text{point})$. An element of $H_2^G(X)$ is represented by a principal G -bundle P over a Riemann surface Σ and a section of the associated bundle $u : \Sigma \rightarrow P(X)$. The $H_2^G(X) \rightarrow H_2^G(\text{point})$ maps the element $[(P, u)]$ to the class $[P] \in H_2^G(\text{point})$. If the domain Σ is connected, it can be covered by two open sets on which P is trivializable. The topology of the bundle $[P]$ is determined by an element $\pi_1(G)$ corresponding to the winding of the transition function of P . By the previous remark, an affine vortex represents a class $\beta \in H_2^G(X)$, and hence there is an induced map

$$(7) \quad \text{hol} : M^G(\mathbb{C}, X) \rightarrow \pi_1(G), \quad [(A, u)] \mapsto [k_0],$$

where $[k_0] \in \pi_1(G)$ is obtained by applying Proposition 2.4 to the affine vortex (A, u) . Strictly speaking, this element is the limit holonomy of the connection A at infinity, but by a slight abuse of terminology, we also call it the *holonomy of the vortex* (A, u) . It is often convenient to use a geodesic representative of the class $[k_0] \in \pi_1(G)$, namely a loop $\theta \mapsto e^{\eta\theta}$ homotopic to k_0 , where $\eta \in \frac{1}{2\pi} \exp^{-1}(\text{Id})$.

Another map associated to an affine vortex is ‘evaluation at infinity’,

$$\text{ev}_\infty : M^G(\mathbb{C}, X) \rightarrow \Phi^{-1}(0)/G,$$

which is well-defined by Proposition 2.4. The maps hol and ev_∞ are continuous under the compact convergence topology on the space of affine vortices $M^G(\mathbb{C}, X)$.

Proposition 2.7 ([25]). *Suppose the sequence of affine vortices (A_i, u_i) converges to a limit (A_∞, u_∞) in the compact convergence topology. Then,*

$$(a) \quad \text{ev}_\infty(A_0, u_0) = \lim_{i \rightarrow \infty} \text{ev}_\infty(A_i, u_i) \text{ and}$$

(b) the class $\text{hol}(A_i, u_i)$ is independent of i and is same as $\text{hol}(A_\infty, u_\infty)$.

Proof of Proposition 2.7. Part (a) is a consequence of convergence of energy of the sequence of vortices and the annulus lemma for vortices [23, Lemma 4.11]. Suppose $X^0 \subset X$ consists of points where the group action is free. Since the images of the vortices are contained in a compact set and X^0 contains an open neighborhood of $\Phi^{-1}(0)$, by uniform asymptotic decay of energy density Proposition 2.2, there exists R_0 such that $u_i(\mathbb{C} \setminus B_{R_0}) \subset X^0$ for all i , including $i = \infty$. The composition of u_i with the projection $X^0 \rightarrow X^0/G$, denoted by u_i/G , is well-defined on $\mathbb{C} \setminus B_{R_0}$. The convergence of energy implies that for any $\epsilon > 0$, there is a large number $R(\epsilon)$ such that

$$(8) \quad E(u_i, \mathbb{C} \setminus B_R) < \epsilon$$

for all i , including $i = \infty$. By the annulus lemma for vortices, there is a constant c independent of ϵ, i such that for any $r_2 > r_1 > 2R$ and any two points in the annulus $z_1, z_2 \in A(r_1, r_2)$,

$$d_{X^0/G}(u_i(z_1)/G, u_i(z_2)/G) \leq cE(u_i, A(r_1/2, 2r_2)) \leq cE(u_i, \mathbb{C} \setminus B_R) \leq c\epsilon.$$

In other words the convergence $u_i(z)/G \rightarrow u_i(\infty)/G$ as $z \rightarrow \infty$ is uniform for all i . Since the maps u_i/G converge to u_∞/G on compact subsets of $\mathbb{C} \setminus B_{R_0}$, we can conclude that $u_i(\infty)/G$ converges to $u_\infty(\infty)/G$.

The proof of part (b) relies on the free group action in the neighborhood of the infinity limits of the maps u_i . By part (a), since the sequence $u_i(\infty)/G$ converges to $u_\infty(\infty)/G$, there is a contractible neighborhood $U \subset X^0/G$ that contains the infinity limits $u_i(\infty)/G$ for large i . The restriction of the principal G -bundle $X^0 \rightarrow X^0/G$ to U is trivializable, and so, there is an equivariant diffeomorphism $\phi_U : U \times G \rightarrow X^0|_U \subset X^0$. For a vortex (A_i, u_i) , the element $\text{hol}(A_i, u_i) \in \pi_1(G)$ is given by considering a loop $[0, 2\pi] \ni \theta \mapsto u_i(R_1 e^{i\theta})$, for $R_1 > R_0$, applying ϕ_U^{-1} and then projecting it to G . Since the maps u_i converge on the circle $\{R_1 e^{i\theta}\}$, $\text{hol}(A_i, u_i)$ is independent of i . \square

2.3. Weighted Sobolev spaces. In this section, we define weighted Sobolev spaces for functions on \mathbb{C} , that are used to define the domain and target space of the vortex differential operator. For $1 < p < \infty$, $\delta \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$ and a compactly supported smooth function $\sigma \in C_0^\infty(\mathbb{C})$, define the following norms:

$$\begin{aligned} \|\sigma\|_{L^{p,\delta}} &:= \|(1 + |z|^2)^{\delta/2} \sigma\|_{L^p} \\ \|\sigma\|_{W^{k,p,\delta}} &:= \sum_{i=0}^k \|\nabla^i \sigma\|_{L^{p,\delta}}, \quad \|\sigma\|_{L^{k,p,\delta}} := \sum_{i=0}^k \|\nabla^i \sigma\|_{L^{p,\delta+i}}. \end{aligned}$$

The spaces $L^{p,\delta}(\mathbb{C})$, $W^{k,p,\delta}(\mathbb{C})$ and $L^{k,p,\delta}(\mathbb{C})$ are the completions of $C_0^\infty(\mathbb{C})$ under the norms $\|\cdot\|_{L^{p,\delta}}$, $\|\cdot\|_{W^{k,p,\delta}}$ and $\|\cdot\|_{L^{k,p,\delta}}$ respectively.

Analogous to Sobolev embedding in the non-weighted case, weighted Sobolev spaces $L^{k,p,\delta}$ embed into a weighted C^0 space. For any $\delta \in \mathbb{R}$, the space $C^{0,\delta}$ is the subspace of continuous functions $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ with finite value of the norm:

$$\|\sigma\|_{C^{0,\delta}} := \sup_{z \in \mathbb{C}} (1 + |z|^2)^{\delta/2} |f(z)|.$$

Lemma 2.8. (Sobolev Embedding for weighted Sobolev spaces, [3, Lemma 2.4]) *For $kp > 2$ and $\delta > \delta' - \frac{2}{p}$, there is a compact embedding $L^{k,p,\delta}(\mathbb{C}) \rightarrow C^{0,\delta'}(\mathbb{C})$.*

Lemma 2.9. (Hardy-type inequality, [23, Proposition E.4]) *Suppose $p > 2$, $\delta > 1 - \frac{2}{p}$ and that $f \in W_{\text{loc}}^{1,p}(\mathbb{C})$ satisfies $\|df\|_{L^{p,\delta}} < \infty$. Then the limit $f(\infty) := \lim_{z \rightarrow \infty} f(z)$ exists and $f - f(\infty) \in L^{p,\delta-1}$ and there is a constant C independent of f such that $\|f - f(\infty)\|_{L^{p,\delta-1}} \leq C\|df\|_{L^{p,\delta}}$. In fact, there is an equivalence of norms:*

$$(9) \quad \|f\|_{L^\infty} + \|\nabla f\|_{L^{p,\delta}} \approx |f(\infty)| + \|f\|_{L^{1,p,\delta-1}}.$$

Proposition 2.10. (Compactness for operators on non-compact domain) *Suppose $\Omega \subset \mathbb{C}$ be a non-compact connected set with smooth boundary. Let $s_1, s_2 \in \mathbb{Z}_{\geq 0}$, $p_1, p_2 > 0$ and $\delta_1, \delta_2 \in \mathbb{R}$. Further, let $F : W^{s_1,p_1,\delta_1}(\Omega) \rightarrow W^{s_2,p_2,\delta_2}(\Omega)$ be a differential operator that satisfies the following. For any compact set $S \subset \Omega$, the restriction $F|_S : W^{s_1,p_1}(S) \rightarrow W^{s_2,p_2}(S)$ is a compact operator. For any R , the restriction $F|_{\Omega \setminus B_R} : W^{s_1,p_1,\delta_1}(\Omega \setminus B_R) \rightarrow W^{s_2,p_2,\delta_2}(\Omega \setminus B_R)$ has bounded norm, and the operator norm $\|F|_{\Omega \setminus B_R}\|$ approaches 0 as $R \rightarrow \infty$. Then, the operator $F : W^{s_1,p_1,\delta_1}(\Omega) \rightarrow W^{s_2,p_2,\delta_2}(\Omega)$ is compact.*

The proof is similar to Lemma 2.1 in [3] and Proposition E.6 (v) in [23], but is reproduced for completeness.

Proof of Proposition 2.10. Suppose σ_i is a bounded sequence in $W^{s_1,p_1,\delta_1}(\Omega)$. We assume $\|\sigma_i\|_{W^{s_1,p_1,\delta_1}(\Omega)} < c$ for all i . By the Banach-Alaoglu theorem, after passing to a subsequence, the sequence σ_i has a weak limit $\sigma_\infty \in W^{s_1,p_1,\delta_1}(\Omega)$. We will show that $F\sigma_i$ strongly converges to $F\sigma_\infty$ in $W^{s_2,p_2,\delta_2}(\mathbb{C})$. Let $\epsilon > 0$ be an arbitrary small number. Choose R such that $\|F|_{\Omega \setminus B_R}\| < \frac{\epsilon}{2c}$. This implies, in $W^{s_2,p_2,\delta_2}(\Omega \setminus B_R)$,

$$(10) \quad \|F\sigma_i - F\sigma_\infty\| < \|F\sigma_i\| + \|F\sigma_\infty\| < \frac{\epsilon}{2}.$$

On the compact set $\Omega \cap B_R$, the compactness of $F|_{\Omega \cap B_R}$ implies that, $F\sigma_i|_{\Omega \cap B_R}$ converges to $F\sigma_\infty|_{\Omega \cap B_R}$. Together with (10), this proves the proposition. \square

The following result is an application of Proposition 2.10.

Lemma 2.11. (Rellich-Kondrachov compactness on weighted Sobolev spaces) *Suppose $k_1 > k_2$, $\delta_1 > \delta_2$ and $p > 1$. The inclusion $W^{k_1,p,\delta_1} \hookrightarrow W^{k_2,p,\delta_2}$ is compact.*

2.4. Outline of proof of main Theorem. Given a vortex (A, u) , we first describe a Banach space parametrizing gauged maps near (A, u) . We define a norm for the elements $(\alpha, \xi) \in \Omega^1(\mathbb{C}, \mathfrak{g}) \times \Gamma(\mathbb{C}, u^*TX)$ as

$$(11) \quad \begin{aligned} \|(\alpha, \xi)\|_{p,\delta} := & \|\xi\|_{L^\infty(\mathbb{C})} + \|\alpha\|_{L^{p,\delta}(\mathbb{C})} + \|\nabla_A \alpha\|_{L^{p,\delta}(\mathbb{C})} + \|\nabla_A \xi\|_{L^{p,\delta}(\mathbb{C})} \\ & + \|d\Phi(\xi)\|_{L^{p,\delta}(\mathbb{C})} + \|d\Phi(J\xi)\|_{L^{p,\delta}(\mathbb{C})}, \end{aligned}$$

where $p > 2$ and $\delta \in (1 - \frac{2}{p}, 1)$. We define a Banach space $\mathcal{B}_{(A,u)}^{p,\delta}$ by

$$\mathcal{B}_{(A,u)}^{p,\delta} := \left\{ (\alpha, \xi) \in W_{\text{loc}}^{1,p}(\mathbb{C}, \Lambda^1 \otimes \mathfrak{g}) \times W_{\text{loc}}^{1,p}(\mathbb{C}, u^*TX) : \begin{aligned} & \|(\alpha, \xi)\|_{p,\delta} < \infty, \\ & d_A^* \alpha + d\Phi(J\xi) = 0 \end{aligned} \right\}.$$

The last condition, which is the vortex Coulomb gauge condition, ensures that the space of vortices cut out of $\mathcal{B}_{(A,u)}^{p,\delta}$ form a slice of the gauge group action. We add a remark about the norm of the ξ term, justifying the lone L^∞ term in (11), while all other norms are $L^{p,\delta}$. This is because we want to allow ξ to have a non-zero limit at infinity - the value of ξ at infinity would correspond to the variation of the evaluation at infinity i.e. $\text{dev}_{\infty,(A,u)}(\alpha, \xi)$. Some equivalent forms of the norm $\|\cdot\|_{p,\delta}$ are discussed later in the paper in Remark 4.7.

In Sections 3, 4 and 5, we prove that vortices in a neighborhood of (A, u) with respect to the compact convergence topology embed continuously into the Banach space $\mathcal{B}_{(A,u)}^{p,\delta}$. That is, if a sequence of affine vortices (A_i, u_i) converge to (A, u) in the compact convergence topology, then, the elements of the sequence can be gauge transformed to a new sequence $(A + \alpha_i, \exp_u \xi_i)$ such that $(\alpha_i, \xi_i) \in \mathcal{B}_{(A,u)}^{p,\delta}$ and $\|(\alpha_i, \xi_i)\|_{p,\delta} \rightarrow 0$ as $i \rightarrow \infty$. This is carried out in 3 steps.

- (a) In Section 3, we show that the connection matrices $A_i - A$ are uniformly bounded in $W^{1,p,\delta}$.
- (b) By an elliptic regularity argument, the bound on connections implies that the $L^{p,\delta}$ -norms of the first derivatives of ξ_i are uniformly bounded in $L^{p,\delta}$. We need another sequence of gauge transformations to ensure an $L^{p,\delta}$ -bound on the term $d\Phi(J\xi_i)$. This is carried out in Section 4.
- (c) As a final step, the vortices in the sequence are gauge transformed so that they are in Coulomb gauge with respect to the limit vortex (A, u) . Then by a bootstrapping argument, we show that the required convergence in the $\|\cdot\|_{p,\delta}$ -norm is satisfied. This step is carried out in Proposition 5.1. The proof of the main result is included in Section 5.

In Section 6, we prove that the vortex differential operator is a Fredholm operator with the expected index. Results related to Coulomb gauge for vortices are presented in Section 7.

3. AN ASYMPTOTIC BOUND ON CONNECTIONS

In this section, we show that for an affine vortex (A, u) , the connection can be expressed in a convenient form, namely outside a disk it is the sum of a flat connection and a term bounded in $W^{1,p,\delta}$. For a converging sequence of vortices, the flat connection can be chosen to be the same for every element of the sequence, and the second term is uniformly bounded in $W^{1,p,\delta}$. This is precisely stated in the following proposition, which is the main result of this section.

Proposition 3.1. (Asymptotic result for connections) *Let $p > 2$, $R > 0$ and $1 - \frac{2}{p} < \delta < 1$. Suppose (A_i, u_i) is a sequence of vortices over $\mathbb{C} \setminus B_R$ that converges to a limit vortex (A_∞, u_∞) in the compact convergence topology. Then there exists a sequence of gauge transformations $g_i \in W_{loc}^{2,p}(\mathbb{C} \setminus B_R, G)$ such that, if we write $g_i A_i = d + a_i$, then $a_i \in W^{1,p,\delta}(\mathbb{C} \setminus B_R)$ and their norms are uniformly bounded for all i .*

The proof of Proposition 3.1 uses Coulomb gauge for connections. Let U be a 2-dimensional compact Riemannian manifold with smooth boundary, and let A_0 be

a smooth reference connection on the trivial bundle $U \times G$. We say that a connection A is in Coulomb gauge with respect to A_0 if

$$d_{A_0}^*(A - A_0) = 0, \quad *(A - A_0)|_{\partial U} = 0.$$

For a connection $A \in L^P$, the boundary trace is not defined. The weak form of the Coulomb gauge condition is

$$\int_U \langle A - A_0, d_{A_0} \psi \rangle = 0 \quad \forall \psi \in C^\infty(U, \mathfrak{g}).$$

Lemma 3.2. (Coulomb gauge for L^P connections, [19, Theorem 8.1, Theorem 8.3]) *Let U be a compact 2-dimensional Riemannian manifold with smooth boundary and $P \rightarrow U$ be a principal G -bundle. Suppose $p > 2$, $k = 0, 1$ and $A_0 \in W^{k,p}$ is a reference connection. Then, there exist constants $\delta > 0$ and $C > 0$ such that for any connection $A \in W^{k,p}$ satisfying $\|A - A_0\|_{W^{k,p}} < \delta$, there is a gauge transformation $g \in W^{k+1,p}(U, G)$ such that gA is in Coulomb gauge with respect to A_0 and*

$$(12) \quad \|gA - A_0\|_{W^{k,p}} \leq C\|A - A_0\|_{W^{k,p}}.$$

The following results are used in the proof of Proposition 3.1.

Lemma 3.3. (Regularity of one-forms) *Let U be a compact 2-dimensional Riemannian manifold with smooth boundary, $P \rightarrow U$ be a principal G -bundle and $p > 2$.*

- (a) *Suppose A_0 is a smooth connection on P , and $A \in L^p$ is in Coulomb gauge with respect to A_0 and its curvature F_A is in $W^{k,p}$ for some $k \in \mathbb{Z}_{\geq 0}$. Then, the connection A is in $W^{k+1,p}$.*
- (b) ([19, Theorem 5.1]) *If a one-form $a \in \Omega^1(U)_{W^{1,p}}$ satisfies $*a|_{\partial U} = 0$, then,*

$$(13) \quad \|a\|_{W^{1,p}(U)} \leq c(\|da\|_{L^p} + \|d^*a\|_{L^p} + \|a\|_{L^p}).$$

Part (a) is a straightforward application of [19, Theorem 5.2] so the proof is omitted.

Proof of Proposition 3.1. We remark that a special case of this proposition is when $A_i = A$ is independent of i . For the general case, all constants involved in the estimates (which come from Propositions 2.2 and 2.4) can be chosen independent of i . To save notations, we carry out the proof in this special case.

We give an outline of the proof. For the vortex (A, u) , after a suitable gauge transformation, the derivative of the connection matrix will have decay property similar to curvature. The connection matrix itself will have an asymptotic L^p bound using (6) of Proposition 2.4. The required gauge transformation is found locally on annuli $A(e^s, e^{s+1})$ and then patched together. We will use the cylindrical metric on $A(e^s, e^{s+1}) \simeq [s, s+1] \times S^1$ to get uniform bounds. In the cylindrical metric, the L^p norm of the curvature is no longer decreasing as we go to infinity. However, the L^p bound of the connection decays in this metric, and so, we use a lower regularity Coulomb gauge result proved by Wehrheim (see Lemma 3.2). The details are as follows.

First, we find the local gauge transformations. Define cylindrical coordinates on $\mathbb{C} \setminus B_R$ as

$$(14) \quad \Psi : [\log R, \infty) \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{C} \setminus B_R, \quad (s, \theta) \mapsto e^{s+i\theta}.$$

Let $k_0 \in W^{1,p}(S^1, G)$ be the limit holonomy of the connection A from Proposition 2.4. We view $k_0 : [\log R, \infty) \times S^1 \rightarrow G$ as a gauge transformation that is independent of the first coordinate. We fix a constant $\frac{1}{2} < \sigma < 1$ and consider bands on the cylinder $S_n := [n\sigma, n\sigma+1] \times S^1$ equipped with the cylindrical metric. In the following discussion, the constants denoted by c are a function of σ , but are independent of n . The L^p norms of the connection matrices $k_0 \Psi^* A - d$ are asymptotically bounded on the bands S_n . By Proposition 2.4, for any $0 < \gamma < \frac{2}{p}$,

$$\|k_0(\Psi^* A) - d\|_{L^p(S_n)} \leq ce^{-\gamma n\sigma}.$$

By Lemma 3.2, for any sufficiently large n , there exists a gauge transformation $g_n \in W^{1,p}(S_n, G)$ that transforms A to Coulomb gauge. That is, if we denote $g_n(\Psi^*(k_0 A)) = d + \tilde{a}_n$, then

$$(15) \quad \int_{S_n} \langle d\psi, \tilde{a}_n \rangle = 0, \quad \forall \psi \in C^\infty(S_n, \mathfrak{g}).$$

By a bootstrapping argument, \tilde{a}_n is smooth. Hence so is $g_n k_0$. Moreover, (12) implies that

$$(16) \quad \|\tilde{a}_n\|_{L^p(S_n)} \leq c\|k_0(\Psi^* A) - d\|_{L^p(S_n)} \leq ce^{-\gamma n\sigma}.$$

A similar computation can be carried out by replacing (p, γ) with $(2p, \gamma/2)$ to obtain

$$(17) \quad \|\tilde{a}_n\|_{L^{2p}(S_n)} \leq ce^{-\gamma n\sigma/2}.$$

The L^p norms of \tilde{a}_n and its derivative behave somewhat differently. By Proposition 2.2, for any $\epsilon > 0$, there is a constant c such that on S_n , $|\Psi^* F_A| < ce^{\epsilon n\sigma}$. By regularity for one-forms (Lemma 3.3 (b)) and the Coulomb gauge condition, we get

$$(18) \quad \begin{aligned} \|\tilde{a}_n\|_{W^{1,p}(S_n)} &\leq c(\|d\tilde{a}_n\|_{L^p} + \|\tilde{a}_n\|_{L^p}) \leq c(\|F_A\|_{L^p(S_n)} + \|\tilde{a}_n\|_{L^{2p}}^2 + \|\tilde{a}_n\|_{L^p}) \\ &\leq c(e^{\epsilon n\sigma} + e^{-\gamma n}) \leq ce^{\epsilon n\sigma}, \end{aligned}$$

where the last two inequalities use the asymptotic curvature bound and the bounds (16) and (17).

Next, we glue together the gauge transformations g_n defined on the bands S_n . Left multiplying g_n by a constant element does not alter the bounds (18) and (16). So, we may assume that for every n , there is a point $x_n \in S_n \cap S_{n+1}$ at which $g_n(x_n) = g_{n+1}(x_n)$. Then, using Lemma 3.4 below and the bounds (18), (16), we can define $\xi_n : S_n \cap S_{n+1} \rightarrow \mathfrak{g}$ by $e^{\xi_n} = g_{n+1}g_n^{-1}$ and it satisfies

$$(19) \quad \|\xi_n\|_{W^{1,p}(S_n \cap S_{n+1})} \leq ce^{-\gamma n\sigma}, \quad \|\xi_n\|_{W^{2,p}(S_n \cap S_{n+1})} \leq ce^{\epsilon n\sigma}.$$

Suppose $\beta_n : S_n \cap S_{n+1} \rightarrow [0, 1]$ be a cut-off function that is 1 in the neighborhood of ∂S_{n+1} and 0 in the neighborhood of ∂S_n . Further, for different values of n , the

functions β_n are translations of each other. We define a gauge transformation \tilde{g} on $\cup_n S_n$ as

$$\tilde{g} := \begin{cases} g_n k_0 & \text{on } S_n \setminus (S_{n-1} \cup S_{n+1}) \\ e^{\beta_n \xi_n} g_n k_0 & \text{on } S_n \cap S_{n+1}. \end{cases}$$

On the intersection $S_n \cap S_{n+1}$, $e^{\beta_n \xi_n}$ satisfies similar bounds as e^{ξ_n} (see (19)). Therefore, after applying the global gauge transformation $\tilde{g} : [\log R, \infty) \times S^1 \rightarrow G$, the connection $d + \tilde{a} := \tilde{g}(\Psi^* A)$ satisfies the same asymptotic bounds as \tilde{a}_n :

$$(20) \quad \|\tilde{a}\|_{W^{1,p}(S_n)} \leq ce^{\epsilon n \sigma}, \quad \|\tilde{a}\|_{L^p(S_n)} \leq ce^{-\gamma n \sigma}.$$

Now, we pull back by Ψ^{-1} to obtain a connection on $\mathbb{C} \setminus B_R$. Under this pullback, the quantities $|\tilde{a}|$ and $|\nabla \tilde{a}|$ get multiplied by different scaling factors. Denoting $a = (\Psi^{-1})^* \tilde{a}$, the estimates in (20) translate to

$$\|a\|_{L^p(\Psi(S_n))} \leq ce^{(-1 + \frac{2}{p} - \gamma)n\sigma}, \quad \|\nabla a\|_{L^p(\Psi(S_n))} \leq ce^{(-2 + \frac{2}{p} + \epsilon)n\sigma}.$$

For any $\delta \in (1 - \frac{2}{p}, 1)$, the connection matrix a is in $W^{1,p,\delta}$, because we can choose $0 < \gamma < \frac{2}{p}$ and $0 < \epsilon < 1 - \frac{2}{p}$ such that

$$\delta - 1 + \frac{2}{p} - \gamma < 0, \quad \delta - 2 + \frac{2}{p} + \epsilon < 0.$$

This finishes the proof of Proposition 3.1. \square

The following lemma used in the proof of Proposition 3.1 is a standard gauge theoretic result (see for example [16, Lemma 1.2]).

Lemma 3.4. *Suppose U be a 2-dimensional compact Riemannian manifold with smooth boundary and $p > 2$. There exist constants $c > 0$ and $\delta > 0$ such that the following is true. Let A_0 be a $W^{1,p}$ -connection on the trivial bundle $U \times G$. For $i = 1, 2$, let $g_i \in W^{2,p}(U, G)$ be gauge transformations satisfying $g_1(x) = g_2(x)$ for some $x \in U$. Let $g_i A_0 = d + a_i$. If $\|a_1\|_{L^p} + \|a_2\|_{L^p} \leq \delta$, then there is a function $\xi : U \rightarrow \mathfrak{g}$ of class $W^{2,p}$ such that $g_1 = e^\xi g_2$ and*

$$\begin{aligned} \|\xi\|_{W^{1,p}} &\leq c(\|a_1\|_{L^p} + \|a_2\|_{L^p}); \\ \|\xi\|_{W^{2,p}} &\leq c(\|a_1\|_{W^{1,p}} + \|a_2\|_{W^{1,p}}). \end{aligned}$$

4. CONVERGENCE OF VORTICES IN A WEIGHTED SOBOLEV NORM

Given a sequence of vortices converging to a limit in the compact convergence topology, in this section, we show that the elements of the sequence can be gauge transformed so that they converge weakly in the (p, δ) -norm defined above. The following Proposition is the main result of the section.

Proposition 4.1. (Weak (p, δ) -convergence for a sequence of vortices) *Let $p > 2$. Suppose (A_i^o, u_i^o) is a sequence of vortices on $\mathbb{C} \setminus B_R$ that converges to a limit (A_∞^o, u_∞^o) in the compact convergence topology. There is a sequence of gauge transformations on \mathbb{C} , $g_i : \mathbb{C} \rightarrow G$ and one for the limit $g_\infty : \mathbb{C} \rightarrow G$ such that the following are satisfied.*

- (a) Let $(A_i, u_i) := g_i(A_i^o, u_i^o)$. There is an element $\eta \in \frac{1}{2\pi} \exp^{-1}(\text{Id})$ such that outside a ball B_R , for all i including $i = \infty$, $A_i = d + \eta d\theta + a_i$ and $a_i \in W^{1,p,\delta}(\mathbb{C} \setminus B_R)$. Further,

$$\|A_i - A_\infty\|_{L^{p,\delta}} + \|\nabla_{A_\infty}(A_i - A_\infty)\|_{L^{p,\delta}} \rightarrow 0 \quad \text{in } L^{p,\delta}(\mathbb{C}).$$

- (b) The limits $u_i(\infty) := \lim_{r \rightarrow \infty} e^{-\eta\theta} u_i(r, \theta)$ exists for all i , including $i = \infty$. The values at infinity converge $\lim_i u_i(\infty) = u_\infty(\infty)$. The sequence $\xi_i := \exp_{u_\infty}^{-1} u_i : \mathbb{C} \rightarrow u_\infty^* TX$ is well-defined and Further,

$$\xi_i \xrightarrow{L^\infty} 0, \quad \nabla_{A_\infty} \xi_i, \nabla_{A_\infty}^2 \xi_i \xrightarrow{L^{p,\delta}} 0.$$

- (c) The sequence $\xi_i : \mathbb{C} \setminus B_R \rightarrow u_\infty^* TX$, defined by $u_i = \exp_{u_\infty} \xi_i$, satisfies $d\Phi(J\xi_i), d\Phi(\xi_i) \rightarrow 0$ in $L^{p,\delta}$.

The technical inputs in the proof of the Proposition are Lemma 4.2, which proves a bound on the derivative of the maps u_i ; and Lemma 4.3, which proves the convergence of the terms $d\Phi(J\xi_i)$ for part (c) of the Proposition. Both these results focus on a complement of the ball in the complex plane. In the first of these results Lemma 4.2, the bound on the derivatives of the maps u_i is a consequence of the uniform asymptotic bounds on the connections proved in Section 3.

Lemma 4.2. (Boundedness of derivatives of maps) *Suppose $R > 0$ and (A_i^o, u_i^o) is a sequence vortices on $\mathbb{C} \setminus B_R$ that converges to a limit (A_∞^o, u_∞^o) in the compact convergence topology. Then, there is a sequence of gauge transformations $g_i : \mathbb{C} \setminus B_R \rightarrow G$ such that the following are satisfied for the sequence $(A_i, u_i) := g_i(A_i^o, u_i^o)$. There exists a large number R_1 such that for all i , including $i = \infty$,*

- (a) *the images $\cup_i u_i(\mathbb{C} \setminus R_1)$ are contained in a chart of X . The maps u_i extend continuously over infinity, and the limits converge in X : $\lim_i u_i(\infty) = u_\infty(\infty)$.*
- (b) *The derivatives $|du_i|$ are uniformly bounded in $W^{1,p,\delta}$ for any $\delta \in (1 - \frac{2}{p}, 1)$.*

In the statement of the above lemma, we make a remark about the definition of the $W^{1,p,\delta}$ norm for the derivatives $|du_i|$. The element du_i is a 1-form taking values in the vector bundle $u_i^* TX$. The $W^{1,p}$ -norm on sections of a vector bundle depend on a choice of connection. We get around this issue by using the fact that the images are contained in a chart and thus, viewing u_i as a \mathbb{C}^n -valued function on $\mathbb{C} \setminus B_{R_1}$. The norm is equivalent for different choices of charts.

Proof. (Proof of Lemma 4.2) We will first prove a uniform bound on the first derivative $|du_i|$ using a bound on connection matrices. By applying Proposition 3.1 to the sequence (A_i^o, u_i^o) , we obtain a sequence of gauge transformations g_i , for which the connection matrices $a_i := g_i A_i^o - d$ are uniformly bounded in $W^{1,p,\delta}$ for all $\delta \in (1 - \frac{2}{p}, 1)$. From now on, we denote $(A_i, u_i) := g_i(A_i^o, u_i^o)$. Since a_i is uniformly bounded in $L^{p,\delta}(\mathbb{C} \setminus B_R)$, the same is true of the vector field $(a_i)_{u_i}$. The twisted derivative $d_{A_i} u_i = du_i + (a_i)_{u_i}$ is uniformly bounded in $L^{p,\delta}(\mathbb{C} \setminus B_R)$ by the uniform constants in Proposition 2.2, which implies that du_i is also uniformly bounded in $L^{p,\delta}(\mathbb{C} \setminus B_R)$. We remark that in a similar way, there is also a uniform L^∞ bound

on du_i . This is because $|a_i|_{L^\infty} < c$ by the Sobolev embedding $W^{1,p,\delta} \subset C^0$, which implies a bound on $|(a_i)_{u_i}|_{L^\infty}$ since the images of u_i are contained in a compact subset of X .

Next we use Hardy's inequality (Lemma 2.9) to show that the maps u_i and u_∞ have limits at infinity. By the definition of the compact convergence topology, there is a compact set $S \subset X$ containing the images of the maps u_i . By Whitney's embedding theorem, S can be embedded into some Euclidean space $i : S \hookrightarrow \mathbb{R}^N$. The compactness of S implies that distances are preserved up to a constant factor, i.e. there is a $c > 0$ such that

$$c^{-1}d_X(x, y) < |i(x) - i(y)| < cd_X(x, y), \quad |di| \leq c, \quad \forall x, y \in S \subset X.$$

Therefore $d(i \circ u_i)$ is uniformly bounded in $L^{p,\delta}$. Since, $i \circ u_i$ maps to Euclidean space, by Hardy's inequality, the maps have limits at infinity, denoted by $u_i(\infty)$ and there is a constant c such that for all i ,

$$(21) \quad \begin{aligned} \|d_X(u_i(z), u_i(\infty))\|_{L^{p,\delta-1}} &\leq c \|du_i\|_{L^{p,\delta}}, \\ \|d_X(u_\infty(z), u_\infty(\infty))\|_{L^{p,\delta-1}} &\leq c \|du_\infty\|_{L^{p,\delta}}. \end{aligned}$$

After applying the projection $\Phi^{-1}(0) \rightarrow X//G$, the infinity limits $u_i(\infty)/G$ converge to $u_\infty(\infty)/G$ in $X//G$, by the continuity of ev_∞ (see Proposition 2.7 (a)). Then, by modifying the gauge transformations g_i by constant factors in G , we can ensure that the values at infinity $u_i(\infty)$ converge to $u_\infty(\infty)$. We continue to call the modified gauge transformations g_i , as a change by a constant factor does not affect any of the preceding analysis. We can now conclude that after increasing R , the images $\cup_i u_i(\mathbb{C} \setminus B_{R_1})$ are contained in a chart of X .

The bound on the second derivative of the maps uses elliptic regularity together with the bound on the derivative of the connection matrices. The calculations assume that u_i maps to \mathbb{R}^{2n} . By holomorphicity of (A_i, u_i) ,

$$\partial_s u_i + J(u_i) \partial_t u_i = -(a_i)_{u_i}^{0,1}$$

Differentiating with respect to s ,

$$(22) \quad (\partial_s + J \partial_t) \partial_s u_i = -(\partial_s J) \partial_t u_i - \partial_s ((a_i)_{u_i}^{0,1}).$$

On any radius 2 disk $B_2(z)$, centered at $z \in \mathbb{C} \setminus B_{R_1}$, the terms in the RHS of (22) can be bounded as

$$\begin{aligned} \|\partial_s (a_i)_{u_i}^{0,1}\|_{L^p(B_2(z))} &\leq c \|du_i\|_{L^\infty} (\|\nabla a_i\|_{L^p(B_2(z))} + \|a_i\|_{L^p(B_2(z))}) \\ &\leq c (\|\nabla a_i\|_{L^p(B_2(z))} + \|a_i\|_{L^p(B_2(z))}); \\ \|(\partial_s J) \partial_t u_i\|_{L^p(B_2(z))} &\leq c \|du_i\|_{L^\infty} \|du_i\|_{L^p(B_2(z))} \leq c \|du_i\|_{L^p(B_2(z))}. \end{aligned}$$

Combining the above two estimates with (22), we have

$$\|(\partial_s u + J \partial_t) \partial_s u_i\|_{L^p(B_2(z))} \leq c (\|a_i\|_{L^p(B_2(z))} + \|\nabla a_i\|_{L^p(B_2(z))} + \|du_i\|_{L^p(B_2(z))}).$$

By elliptic regularity (see Proposition B.4.9 in [11]),

$$(23) \quad \|\partial_s u_i\|_{W^{1,p}(B_1(z))} \leq (\|a_i\|_{L^p(B_2(z))} + \|\nabla a_i\|_{L^p(B_2(z))} + \|du_i\|_{L^p(B_2(z))}).$$

A similar statement is true for $\partial_t u_i$ also. Since a_i , ∇a_i and du_i are in $L^{p,\delta}$, we can conclude that $\nabla^2 u_i \in L^{p,\delta}(\mathbb{C} \setminus B_R)$, and the norm is independent of i . This bound

can be obtained by multiplying the bound (23) on $B_1(z)$ by $|z|^\delta$ and adding the inequalities for all the disks $B_1(z)$, where the sum ranges over $z \in (\mathbb{Z} + i\mathbb{Z}) \cap (\mathbb{C} \setminus B_{R_1})$ in the integer lattice. \square

For a converging sequence of vortices, we have chosen gauges for each element outside a ball, such that the connection matrices and the the derivatives of the maps are uniformly bounded. We next apply another sequence of gauge transformations to the sequence of vortices so that the quantities $d\Phi(J\xi_i)$ converge to 0 in $L^{p,\delta}$ as $i \rightarrow \infty$. Here $\xi_i := \exp_{u_\infty}^{-1} u_i$.

Lemma 4.3. (Gauge-fixing relative to limit vortex) *Suppose (A_i, u_i) is a sequence of vortices and (A_∞, u_∞) is another vortex, all of which are defined on $\mathbb{C} \setminus B_R$. The following are satisfied for any $p > 2$ and $\delta \in (1 - \frac{2}{p}, 1)$: for all i , including $i = \infty$,*

- (a) *the connection matrices $a_i := A_i - d$ are uniformly bounded in $W^{1,p,\delta}(\mathbb{C} \setminus B_R)$.*
- (b) *The images of the maps $u_i : \mathbb{C} \setminus B_R \rightarrow X$ are contained in a coordinate chart of X . The derivatives du_i are bounded in $W^{1,p,\delta}$.*
- (c) *The maps $u_i : \mathbb{C} \setminus B_R \rightarrow X$ extend continuously over infinity. The infinity limits $u_i(\infty)$ converge to $u_\infty(\infty)$.*

Then, there is a sequence of gauge transformations $g_i : \mathbb{C} \setminus B_R \rightarrow G$ such that the properties (a)-(c) continue to be satisfied for the sequence $g_i(A_i, u_i)$, and in addition if $g_i u_i = \exp_{u_\infty} \xi_i$, then $d\Phi_{u_\infty}(J\xi_i) = 0$ for all i .

Proof. The result of the Lemma, that $d\Phi(J\xi_i)$ vanishes, is a pointwise gauge-fixing condition for the vortices in the sequence with respect to the limit vortex. In particular, for a point $z \in \mathbb{C}$, if the G -action is free on $u_\infty(z)$, a slice of the action is given by $\exp_{u_\infty(z)}(\ker d\Phi \circ J)$. So, there exist gauge transformations $g_i : \mathbb{C} \setminus B_R \rightarrow G$ such that the term $d\Phi_{u_\infty}(J\xi_i)$ vanishes for all i .

We next construct the gauge transformations in a more rigorous way, and show that the asymptotic properties of g_i are similar to u_i , and hence properties (a)-(c) are satisfied for the sequence $g_i(A_i, u_i)$. By increasing R , we can assume that the G -action is free on the image of the maps u_i . Let X^0 be a compact G -invariant neighborhood in X on which the G action is free and which contains the images of the maps u_i . At any point $x \in X^0$, the tangent space splits into the tangent space of the G -orbit and its orthogonal complement: $T_x X = \mathfrak{g}_x \oplus \mathfrak{g}_x^\perp$. This induces a splitting of the tangent bundle: $TX|_{X^0} := \mathfrak{g}_{X^0} \oplus \mathfrak{g}_{X^0}^\perp$. The restriction of the map

$$\mathcal{L} : \mathfrak{g}_{X^0}^\perp \times \mathfrak{g} \rightarrow X^0 \times X, \quad ((x, \xi), \zeta) \mapsto (x, e^\zeta \exp_x \xi)$$

to a neighborhood of the zero section in $\mathfrak{g}_{X^0}^\perp$ and a neighborhood of the origin in \mathfrak{g} is a diffeomorphism onto its image. The image is a neighborhood of the diagonal $\Delta(X^0)$, which we denote by $N(\Delta(X^0))$. We work with the inverse of \mathcal{L} composed with projection to \mathfrak{g} , which we denote by $\mathcal{L}_\mathfrak{g}^{-1}$, namely

$$\mathcal{L}_\mathfrak{g}^{-1} : N(\Delta(X^0)) \rightarrow \mathfrak{g}.$$

By increasing R and dropping some initial terms of the sequence, we may assume the image of $(u_\infty, u_i)|_{\mathbb{C} \setminus B_R}$ is contained in $N(\Delta(X^0))$. We can then define $\zeta_i : \mathbb{C} \setminus B_R \rightarrow \mathfrak{g}$ as $\zeta_i(z) := \mathcal{L}_\mathfrak{g}^{-1}(u_\infty(z), u_i(z))$. The required gauge transformations g_i are then $e^{-\zeta_i}$.

It is easily seen that $d\Phi_{u_\infty}(J\zeta_i) = 0$ on $\mathbb{C} \setminus B_R$. The derivatives of \mathcal{L}_g^{-1} are bounded and so, the asymptotic behavior of ζ_i is same as u_i . In particular $d\zeta_i$ is bounded in $W^{1,p,\delta}(\mathbb{C} \setminus B_R)$. Therefore, the properties (a)-(c) are satisfied for the sequence $e^{-\zeta_i}(A_i, u_i)$, finishing the proof. \square

The two technical ingredients Lemma 4.2 and Lemma 4.3 can be combined to give the following result on the complement of a ball. It goes one step forward; whereas Lemmas 4.2 and 4.3 show uniform boundedness under the p, δ -norm, the following Lemma shows the existence of a unique weak limit.

Lemma 4.4. (A weak (p, δ) -limit outside a ball) *Let $p > 2$. Suppose (A_i^o, u_i^o) is a sequence of vortices on $\mathbb{C} \setminus B_R$ that converges to a limit (A_∞^o, u_∞^o) in the compact convergence topology. There is a large number R and gauge transformations $g_i : \mathbb{C} \setminus B_R \rightarrow G$ for all i including $i = \infty$ such that the following are satisfied:*

- (a) $g_i A_i^o = d + a_i$, and $a_i \in W^{1,p,\delta}(\Omega^1(\mathbb{C}, \mathfrak{g}))$ for any $\delta \in (1 - \frac{2}{p}, 1)$. Further, the sequence a_i converges to a_∞ weakly in $W^{1,p,\delta}$.
- (b) The maps $u_i := g_i u_i^o$ and $u_\infty := g_\infty u_\infty^o$ have limits at infinity, denoted by $u_i(\infty)$ and $u_\infty(\infty)$ respectively. The sequence of limits $u_i(\infty)$ converge to $u_\infty(\infty)$. The images $u_i(\mathbb{C} \setminus B_R)$ are contained in a chart of X . The sequence du_i converges to du_∞ weakly in $W^{1,p,\delta}$ for any $\delta \in (1 - \frac{2}{p}, 1)$, and the sequence u_i converges to u_∞ in $L^\infty(\mathbb{C} \setminus B_R)$.
- (c) The sequence $\xi_i : \mathbb{C} \setminus B_R \rightarrow u_\infty^* TX$, defined by $u_i = \exp_{u_\infty} \xi_i$, satisfies $d\Phi(J\xi_i) = 0$ and $d\Phi(\xi_i) \rightarrow 0$ in $L^{p,\delta}$.
- (d) The gauge transformations g_i converge to g_∞ weakly in $W^{2,p}$ in compact subsets of $\mathbb{C} \setminus B_R$.

Proof. (Proof of Lemma 4.4) By applying Lemma 4.2, followed by Lemma 4.3 to the sequence of vortices (A_i^o, u_i^o) on $\mathbb{C} \setminus B_R$, we get a sequence of gauge transformations $g_i : \mathbb{C} \setminus B_R \rightarrow G$ for all i including $i = \infty$, such that the results of Lemma 4.4 are partially satisfied. In particular, we know that the sequences A_i, du_i are bounded in $W^{1,p,\delta}$, which implies that there is a weakly converging subsequence. We need to prove that any converging subsequence has the same limit. This will follow if we prove part (d) of Lemma 4.4, which says that the sequence of gauge transformations converges to g_∞ on compact subsets. In fact, that would force the weak limit of the vortices $g_i(A_i^o, u_i^o)$ to be $g_\infty(A_\infty^o, u_\infty^o)$.

We now prove part (d) of the Lemma. On any compact subset $S \subset \mathbb{C} \setminus B_R$, the sequence of gauge transformations g_i is bounded in $W^{2,p}(S)$, because both sequences of connections A_i^o and $A_i := g_i A_i^o$ are bounded in $W^{1,p}(S)$. Therefore, a subsequence of g_i has a weak $W^{2,p}(S)$ limit, which we denote g'_∞ . We will now prove that $g_\infty = g'_\infty$ on S . Let X^0 be a neighborhood of $\Phi^{-1}(0)$ on which the G -action is free. By increasing R , we can assume that the images $u_i(\mathbb{C} \setminus B_R)$ are contained in X^0 . The weak convergence $g_i \rightharpoonup g'_\infty$ in $W^{2,p}(S)$ implies a strong convergence in $C^0(S)$. Together with the compact convergence of u_i^o to u_∞^o , we can conclude that the sequence $g_i u_i^o$, which is same as u_i , converges to $g'_\infty u_\infty^o$ in $C^0(S)$. For any point $z \in S$, the condition $d\Phi(J\xi_i(z)) = 0$ implies that the points $u_i(z)$ lie on $\exp_{u_\infty(z)}(\ker(d\Phi(J \cdot)))$, which is a slice of the G -action at $u_\infty(z)$. Therefore the limit

of the points $u_i(z)$ is also in the slice, and is just $u_\infty(z)$. Therefore u_∞ , which was defined as $g_\infty u_\infty^o$, coincides with $g'_\infty u_\infty^o$. Since the G -action is free at $u_\infty(z)$, we can conclude $g_\infty = g'_\infty$. This finishes the proofs of parts (a), (b) and (d) of the Lemma. In part (c), it follows from the conclusion of Lemma 4.3 that $d\Phi(\xi_i) = 0$. The convergence $d\Phi(\xi_i) \rightarrow 0$ is a consequence of Lemma 4.5 given below. \square

The following Lemma was used in the proof of Lemma 4.4.

Lemma 4.5. *Suppose (A_i, u_i) is a sequence of vortices that converges to a limit vortex (A_∞, u_∞) in the compact convergence topology. Further, suppose u_i converges to u_∞ in $L^\infty_{\text{loc}}(\mathbb{C})$. Then, for any $p > 2$ and $\delta < 1 - \frac{2}{p}$,*

- (a) *the sequence $\Phi(u_i)$ converges to $\Phi(u_\infty)$ in $L^{p,\delta}$.*
- (b) *The sequence $d\Phi(\xi_i)$ converges to 0 in $L^{p,\delta}$, where $\xi_i := \exp_{u_\infty}^{-1} u_i$.*

Proof. The convergence is proved using asymptotic decay estimate for outside a compact set and the L^∞ estimate inside a compact set. By the uniform constants in asymptotic decay for vortices, Proposition 2.2, for any $\nu > 0$, there is a constant c such that

$$|\Phi(u_i(z))| \leq c|z|^{-2+\nu} \forall z \in \mathbb{C}, i.$$

Fix $\nu < 2 - \frac{2}{p} - \delta$ and let $\gamma := 2 - \frac{2}{p} - \delta - \nu$. Therefore, for any R and i ,

$$(24) \quad \|\Phi(u_i) - \Phi(u_\infty)\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} \leq \|\Phi(u_i)\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} + \|\Phi(u_\infty)\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} \leq cR^{-\gamma}.$$

Given $\epsilon > 0$, by (24), R can be chosen so that $\|\Phi(u_i) - \Phi(u_\infty)\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} < \frac{\epsilon}{2}$. Inside the ball B_R , using L^∞ convergence of u_i , for large enough i , $\|\Phi(u_i) - \Phi(u_\infty)\|_{L^{p,\delta}(B_R)} < \frac{\epsilon}{2}$, which proves part (a).

For the proof of part (b), consider a small neighborhood S of $\Phi^{-1}(0)$. There exists constants c_1, c_2 such that for all $x \in S$, $v \in T_x X$ satisfying $|v| < c_1$, we can say $|d\Phi_x(v)| \leq c_2(|\Phi(\exp_x v) - \Phi(x)|)$. For a large R , we can assume the images $u_i(\mathbb{C} \setminus B_R)$ are contained in S . For large i , $\|\xi_i\|_{L^\infty} < c_1$. Then, for $z \in \mathbb{C} \setminus B_R$,

$$|d\Phi(\xi_i)| \leq c_2(|\Phi(u_\infty(z))| + |\Phi(u_i(z))|) \leq c|z|^{-2+\nu},$$

where the last inequality follows from the uniform asymptotic decay of energy density, Proposition 2.2. Then,

$$(25) \quad \|d\Phi(\xi_i)\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} \leq cR^{-\gamma},$$

where $\gamma := 2 - \frac{2}{p} - \delta - \nu$. The inequality (25) is analogous to (24). The proof of part (b) is completed in the same way as part (a) using the fact that ξ_i converges to 0 in $L^\infty(\mathbb{C})$. \square

Having proved a (p, δ) -weak convergence result outside the ball, we are ready to prove the main result of the section, which is a similar convergence result on all of \mathbb{C} .

Proof. (Proof of Proposition 4.1) On the given sequence of vortices, we first apply the p, δ -weak convergence result outside a ball. By applying Lemma 4.4 on the sequence of vortices (A_i^o, u_i^o) , we get a sequence of gauge transformations $g'_i : \mathbb{C} \setminus B_R \rightarrow G$, for some $R > 0$. These gauge transformations may not extend to all of \mathbb{C} for topological

reasons. However, after gauge transforming by a uniform twist (that is independent of i), they extend over \mathbb{C} . The convergence in the Proposition then extends to analogous convergence results over all of \mathbb{C} , which proves Proposition 4.1, the main result of the section.

We first claim that the homotopy type of the sequence of gauge transformations is independent of i , if i is large enough. The vortices $g'_i(A_i^o, u_i^o)$, defined on $\mathbb{C} \setminus B_{2R}$ have trivial holonomy, because u_i^o extends continuously over infinity. On the other hand, for the vortices (A_i, u_i) , the holonomy $\text{hol}(A_i, u_i) \in \pi_1(G)$ is not necessarily trivial. However, this quantity is independent of i for large i , see Proposition 2.7 (b). Therefore, the homotopy type of the gauge transformations $g'_i : \mathbb{C} \setminus B_R \rightarrow G$, given by $-\text{hol}(A_i, u_i)$, is independent of i .

The corollary is proved by applying a constant twist to the sequence of gauge transformations defined on the complement of a ball, and then extending them inside the ball. We can choose an element $\eta \in \frac{1}{2\pi} \exp^{-1}(\text{Id})$ such that $\text{hol}(A_i, u_i) \in \pi_1(G)$ is represented by the geodesic loop $[0, \pi] \ni \theta \mapsto e^{\eta\theta}$. Then, the map $e^{-\eta\theta} g'_i : \mathbb{C} \setminus B_R \rightarrow G$ is contractible. Using a cut-off function, we construct gauge transformations $g_i : \mathbb{C} \rightarrow G$ such that $g_i = e^{-\eta\theta} g'_i$ on $\mathbb{C} \setminus B_{2R}$ and $g_i = \text{Id}$ on B_R .

We claim that the vortices $(A_i, u_i) := g_i(A_i^o, u_i^o)$, now defined over \mathbb{C} , satisfy the claims in the Corollary. The sequence converges to the limit (A_∞, u_∞) weakly in $W^{1,p} \times W^{2,p}$ in compact subsets. This follows from the weak convergence of the sequences (A_i^o, u_i^o) and g_i on compact subsets. In the complement of the ball, writing $\alpha_i = A_i - A_\infty$, by Lemma 4.6 (a) below there is an equivalence of norms

$$\|\text{Ad}_{e^{-\eta\theta}} \alpha_i\|_{W^{1,p,\delta}(\mathbb{C} \setminus B_R)} \approx \|\alpha_i\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} + \|\nabla_{A_\infty} \alpha_i\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)},$$

which proves part (a) of Proposition 4.1. We know that by assuming R to be large enough, the images $e^{-\eta\theta} u_i(\mathbb{C} \setminus B_R)$ is contained in a chart of X . We denote $u_i^\eta := e^{-\eta\theta} u_i$ and $\xi_i^\eta := e^{-\eta\theta} \xi_i \in \Gamma(\mathbb{C} \setminus B_R, (u_i^\eta)^* TX)$. The convergences in part (b) of Lemma 4.4 can be re-written as

$$\xi_i^\eta \xrightarrow{L^\infty(\mathbb{C} \setminus B_R)} 0, \quad \nabla_{\xi_i^\eta} \xrightarrow{W^{1,p,\delta}(\mathbb{C} \setminus B_R)} 0.$$

By Lemma 4.6 (b) below, the above convergence is equivalent to the convergence in part (b) of the Proposition. Part (c) follows from the corresponding result Lemma 4.4 (c) and the L^∞ convergence of u_i . \square

In the last part of this section (Lemma 4.6 and Remark 4.7), we discuss some equivalent forms of certain weighted Sobolev norms for sections on \mathbb{C} .

Lemma 4.6. (Equivalence of norms of covariant derivatives) *Let $p > 2$ and $\delta \in (1 - \frac{2}{p}, 1)$. Suppose (A, u) is a vortex on the trivial bundle $P = (\mathbb{C} \setminus B_R) \times G$, and $A = d + a$ where $a \in W^{1,p,\delta}(\Omega^1(\mathbb{C} \setminus B_R, \mathfrak{g}))$.*

- (a) *Let E be an associated vector bundle of P . For $k = 1, 2$, the $W^{k,p,\delta}$ norm on sections $\sigma \in \Gamma(\mathbb{C}, E)$ defined using the trivial connection on E is equivalent*

to the one defined using the connection A . That is,

$$(26) \quad \sum_{i=0}^k \|\nabla^i \sigma\|_{L^{p,\delta}} \approx \sum_{i=0}^k \|\nabla_A^i \sigma\|_{L^{p,\delta}}.$$

(b) Suppose the image $u(\mathbb{C} \setminus B_R)$ is contained in a chart of X . For sections $\xi : \mathbb{C} \setminus B_R \rightarrow u^*TX$, there is an equivalence of norms:

$$(27) \quad \|\xi\|_{L^\infty} + \sum_{i=1}^k \|\nabla^i \xi\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} \approx \|\xi\|_{L^\infty} + \sum_{i=1}^k \|\nabla_A^{X,i} \xi\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)}.$$

Proof. We denote the norm in the lhs of (26) by $\|\cdot\|_{W_A^{k,p,\delta}}$. We prove the equivalence by induction. Assuming the statement is true for $k-1$. Then,

$$\begin{aligned} \|\sigma\|_{W_A^{k,p,\delta}} &= \|\sigma\|_{L^{p,\delta}} + \|\nabla_A \sigma\|_{W_A^{k-1,p,\delta}} = \|\sigma\|_{L^{p,\delta}} + \|\nabla_A \sigma\|_{W^{k-1,p,\delta}} \\ &\leq \|\sigma\|_{L^{p,\delta}} + \|\nabla \sigma\|_{W_A^{k-1,p,\delta}} + \|[a, \sigma]\|_{W^{k-1,p,\delta}}. \end{aligned}$$

If $k=1$, the last term is bounded as $\|[a, \sigma]\|_{L^{p,\delta}} \leq c\|\sigma\|_{L^{p,\delta}}$, using the L^∞ bound on a coming from the inclusion $W^{1,p,\delta} \hookrightarrow L^\infty$ (see Lemma 2.8). If $k=2$, we have

$$\|[a, \sigma]\|_{W^{1,p,\delta}} = \|[a, \sigma]\| + \|[\nabla a, \sigma]\| + \|[a, \nabla \sigma]\|_{L^{p,\delta}} \leq c\|a\|_{W^{1,p,\delta}} \|\sigma\|_{W^{1,p,\delta}},$$

where in the last inequality, the second and third term are bounded by applying the embedding $W^{1,p,\delta} \hookrightarrow L^\infty$ to σ and a respectively. We have thus proved $\|\sigma\|_{W_A^{k,p,\delta}} \leq c\|\sigma\|_{W^{k,p,\delta}}$. The reverse inequality can be proved similarly, which finishes the proof of the equivalence (26).

The second part of the lemma is proved similarly and uses the fact that du is bounded in $W^{1,p,\delta}$ (as in the proof of Lemma 4.2). Let $U \subset X$ be the chart containing the image of the map u . We recall that $\nabla_A^X \xi := \nabla \xi + (a^X \circ du)\xi + \nabla_\xi^X a_X$, where $a^X \in \Omega^1(U, \text{End}(TX))$ is the connection matrix for the Levi-Civita connection. The lower order terms $(a^X \circ du)\xi$ (resp. $\nabla_\xi^X a_X$) are linear in ξ and du (resp. a), and can be viewed as a family of bilinear operators parametrized smoothly by $\mathbb{C} \setminus B_R$. The norm of this bilinear operator and its derivative is L^∞ -bounded. Therefore, the equivalence of norms (27) can be proved in the same way as (26). \square

Remark 4.7. (Equivalent forms of the (p, δ) norm) Suppose (A, u) is a finite energy affine vortex whose image is bounded. Recall the (p, δ) norm for deformations of the vortex $(\alpha, \xi) \in \Omega^1(\mathbb{C}, u^*TX) \times \Gamma(\mathbb{C}, u^*TX)$:

$$\begin{aligned} \|(\alpha, \xi)\|_{p,\delta} &:= \|\xi\|_{L^\infty(\mathbb{C})} + \|\alpha\|_{L^{p,\delta}(\mathbb{C})} + \|\nabla_A \alpha\|_{L^{p,\delta}(\mathbb{C})} \\ &\quad + \|\nabla_A \xi\|_{L^{p,\delta}(\mathbb{C})} + \|d\Phi(\xi)\|_{L^{p,\delta}(\mathbb{C})} + \|d\Phi(J\xi)\|_{L^{p,\delta}(\mathbb{C})}. \end{aligned}$$

The definition of the (p, δ) norm is gauge-invariant. After choosing a gauge and some other quantities, the norm can be stated in an equivalent form using Lemma 4.6. After gauge transforming on $\mathbb{C} \setminus B_R$, for some $R > 0$, the connection A is equal to $d + \eta d\theta + a$, where $\eta \in \frac{1}{2\pi} \exp^{-1}(\text{Id})$ and $a \in W^{1,p,\delta}$. By Lemma 4.6, there is an equivalence of norms:

$$\|\alpha\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} + \|d_A \alpha\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)} \sim \|\alpha\|_{W^{1,p,\delta}(\mathbb{C} \setminus B_R)}.$$

For examining the deformation of the map u , given by ξ , we work with a twisted version of the map. We recall that the map $e^{-\eta\theta}u$ extends continuously over infinity. Then, $\nabla_A \xi \in L^{p,\delta}$ is equivalent to $\nabla(e^{-\eta\theta}\xi) \in L^{p,\delta}$, where $e^{-\eta\theta}\xi$ is a section in $\Gamma(\mathbb{C} \setminus B_R, (e^{-\eta\theta}u)^*TX)$. Suppose R be large enough that $e^{-\eta\theta}u(\mathbb{C} \setminus B_R)$ is contained in a chart of X . Then, $e^{-\eta\theta}\xi$ can be viewed as a vector space valued function, whose derivative is in $L^{p,\delta}(\mathbb{C} \setminus B_R)$. By Hardy's inequality (Lemma 2.9), at infinity $e^{-\eta\theta}\xi$ has a limit $(e^{-\eta\theta}\xi)(\infty) \in X$. Choose positive constants $R_1 < R_2$. There is an equivalence of norms

$$\|\xi\|_{L^\infty(\mathbb{C})} + \|\nabla_A \xi\|_{L^{p,\delta}(\mathbb{C})} \sim |(e^{-\eta\theta}\xi)(\infty)| + \|d(e^{-\eta\theta}\xi)\|_{L^{p,\delta}(\mathbb{C} \setminus B_{R_1})} + \|\xi\|_{W^{1,p}(B_{R_2})}.$$

5. CONSTRUCTION OF LOCAL MODEL

Recall from Section 2.4 that for a given vortex (A, u) , we defined a Banach space structure for the space of gauged maps in the neighborhood of (A, u) . In this neighborhood, we defined a subspace $\mathcal{B}_{(A,u)}^{p,\delta}$ consisting of gauged maps that are in Coulomb gauge with respect to (A, u) . In the following Proposition, we prove that the compact convergence topology on vortices coincides with convergence in the Banach space $\mathcal{B}_{(A,u)}^{p,\delta}$. Using this result, the main Theorem is proved at the end of this section.

Proposition 5.1. *Let $p > 2$ and $\delta \in (1 - \frac{2}{p}, 1)$. Suppose (A_i^o, u_i^o) is a sequence of affine vortices converging to a limit (A_∞, u_∞) in the compact convergence topology. Then, there is a sequence of gauge transformations $g_i : \mathbb{C} \rightarrow G$ such that if $(A_i, u_i) := g_i(A_i^o, u_i^o)$ and $\alpha_i := A_i - A_\infty$, $u_i = \exp_{u_\infty} \xi_i$ then, for large i , $(\alpha_i, \xi_i) \in \mathcal{B}_{(A_\infty, u_\infty)}^{p,\delta}$. Further, $\|(\alpha_i, \xi_i)\|_{p,\delta} \rightarrow 0$ as $i \rightarrow \infty$.*

Proof. We first obtain a weak convergence result by applying Proposition 4.1 to the sequence of vortices (A_i^o, u_i^o) and their limit (A_∞^o, u_∞^o) . The result is a sequence of gauge transformations h_i and a gauge transformation h_∞ for the limit, such that the transformed sequence $(A'_i, u'_i) := h_i(A_i^o, u_i^o)$ has a weak limit $(A_\infty, u_\infty) := h_\infty(A_\infty^o, u_\infty^o)$ in the following sense:

$$(28) \quad \begin{aligned} A'_i - A_\infty &\xrightarrow{W^{1,p,\delta_1}} 0, & u'_i &\xrightarrow{L^\infty} u_\infty, & d_{A_\infty} \xi'_i, \nabla_{A_\infty} (d_{A_\infty} \xi'_i) &\xrightarrow{L^{p,\delta_1}} 0, \\ d\Phi(\xi'_i) &\xrightarrow{L^{p,\delta}} 0, & d\Phi(J\xi'_i) &\xrightarrow{L^{p,\delta}} 0, \end{aligned}$$

where $\delta_1 \in (\delta, 1)$ and $\xi'_i := \exp_{u_\infty}^{-1} u'_i : \mathbb{C} \rightarrow u_\infty^*TX$. Further on $\mathbb{C} \setminus B_R$, for any R , the limit connection A_∞ is of the form $A_\infty = d + \eta d\theta + a_\infty$ where $\eta \in \frac{1}{2\pi} \exp^{-1}(\text{Id}) \subset \mathfrak{g}$ and $a \in W^{1,p,\delta_1}(\Omega^1(\mathbb{C} \setminus B_R, \mathfrak{g}))$. The proposition requires strong convergence, so we shift to working with lower regularity Sobolev spaces. The weak convergence in (28), together the compact Kondrachov-Rellich inclusion (Lemma 2.11) implies

$$(29) \quad A'_i - A_\infty \xrightarrow{L^{p,\delta}} 0, \quad d_{A_\infty} \xi'_i \xrightarrow{L^{p,\delta}} 0.$$

For the term $d_{A_\infty} \xi'_i$, we also use Lemma 4.6 to say that $d_{A_\infty} \xi'_i$ weakly converges in W^{1,p,δ_1} .

Next, we gauge transform so that vortices in the sequence are in Coulomb gauge with respect to the limit vortex. By (28) and (29), the sequence $d_{A_\infty}^* \alpha'_i + d\Phi(J\xi'_i)$ converges to 0 in $W^{-1,p,\delta}(\mathbb{C})$. Therefore, for large enough i , by Proposition 7.1, there is a sequence of gauge transformations e^{s_i} that put (A'_i, u'_i) in Coulomb gauge with respect to (A_∞, u_∞) . The sequence s_i converges to 0 in $W^{1,p,\delta}$. We denote $(A_i, u_i) := e^{s_i}(A'_i, u'_i)$ and define ξ_i as satisfying $u_i = \exp_{u_\infty} \xi_i$. The following convergence properties hold for the new sequence (A_i, u_i) :

$$(30) \quad \begin{aligned} A_i - A_\infty &\xrightarrow{L^{p,\delta}} 0, & u_i &\xrightarrow{L^\infty} u_\infty, & d_{A_\infty} \xi_i &\xrightarrow{L^{p,\delta}} 0, \\ d\Phi(\xi'_i) &\xrightarrow{L^{p,\delta}} 0, & d\Phi(J\xi'_i) &\xrightarrow{L^{p,\delta}} 0. \end{aligned}$$

To finish the proof, we only need to show the convergence of connections in a higher regularity space. Currently, $\alpha_i := A_i - A_\infty$ converges in $L^{p,\delta}$. We additionally require the convergence of $\nabla_{A_\infty} \alpha_i$ in $L^{p,\delta}$. We prove this separately for inside and outside a large ball B_R . The radius of the ball R is chosen so that the connection matrix a_∞ satisfies $\|a_\infty\|_{L^\infty(\mathbb{C} \setminus B_R)} < c_1$, where c_1 is from Lemma 5.2. In addition to the convergences in (30), we also have

$$F(A_i) \xrightarrow{L^{p,\delta}} F(A_\infty), \quad d_{A_\infty}^* \alpha_i \xrightarrow{L^{p,\delta}} 0.$$

The first convergence is a consequence of the vortex equation and from the convergence of the series $\Phi(u_i) \rightarrow \Phi(u_\infty)$ in $L^{p,\delta}$ using Lemma 4.5. The convergence of $d_{A_\infty}^* \alpha_i$ is a result of the Coulomb gauge condition and from the convergence of $d\Phi(J\xi'_i)$. Using elliptic bootstrapping result for connections, Lemma 5.2, it follows that $\nabla_{A_\infty} \alpha_i$ converges to 0 in $L^p(B_R)$. To analyze the region outside the ball, we cover it with identical disks $B_1(z)$, where z is in the integer lattice $(\mathbb{Z} + i\mathbb{Z}) \cap (\mathbb{C} \setminus B_R)$. We twist all the quantities by the gauge transformation $e^{-\eta\theta} : \mathbb{C} \setminus B_R \rightarrow G$ and denote $\alpha_i^\eta := \text{Ad}_{e^{-\eta\theta}} \alpha_i$, $A_\infty^\eta := e^{-\eta\theta} A_\infty = d + a_\infty^\eta$ etc. Applying Lemma 5.2 (given below) to the sequence of connections A_i^η on each tile, we get

$$(31) \quad \begin{aligned} \|\nabla_{A_\infty} \alpha_i\|_{L^p(B_1(z))} &\leq c(\|\alpha_i\|_{L^p(B_2(z))} + \|F(A_i) - F(A_\infty)\|_{L^p(B_2(z))} \\ &\quad + \|d_{A_\infty}^* \alpha_i\|_{L^p(B_2(z))}), \end{aligned}$$

where the constant c is independent of z . The quantities in the RHS of (31) converge to zero in $L^{p,\delta}$ as $i \rightarrow \infty$. Therefore, by scaling each inequality by $|z|^\delta$ and adding all of them, we can conclude that $\nabla_{A_\infty} \alpha_i$ converges to zero in $L^{p,\delta}(\mathbb{C} \setminus B_R)$. \square

Lemma 5.2. (Elliptic bootstrapping for connections) *Suppose Ω', Ω are compact sets with smooth boundary in \mathbb{C} , such that $\Omega' \subset \text{int}(\Omega)$. Let $p > 2$ and $\delta \in (1 - \frac{2}{p}, 1)$. Then there exist positive constants ϵ , c_1 and c such that following holds. Suppose $A_i = d + a_i$ is a sequence of connections on the trivial bundle $\Omega \times G$ and a limit connection $A_\infty = d + a_\infty$, such that in Ω ,*

$$F_{A_i} \xrightarrow{L^p} F_{A_\infty}, \quad A_i \xrightarrow{L^p} A_\infty, \quad d_{A_\infty}^* (A_i - A_\infty) \xrightarrow{L^p} 0.$$

Then, the sequence A_i converges to A_∞ in $W^{1,p}(\Omega')$. Further if

$$\|A_i - A_\infty\|_{L^p(\Omega)}, \|F(A_i) - F(A_\infty)\|_{L^p(\Omega)}, \|d_{A_\infty}^* (A_i - A_\infty)\|_{L^p(\Omega)} < \epsilon, \|a_\infty\|_{C^0} < c_1.$$

then,

$$(32) \quad \|\nabla_{A_\infty}(A_i - A_\infty)\|_{L^p(\Omega')} \leq c(\|A_i - A_\infty\|_{L^p(\Omega)} + \|F(A_i) - F(A_\infty)\|_{L^p(\Omega)} + \|d_{A_\infty}^*(A_i - A_\infty)\|_{L^p(\Omega)})$$

Proof. The proof is by elliptic regularity of the operator $d \oplus d^*$ on one-forms. The bound on the d^* term is given and the bound on the d will be produced from the curvature bound. The details are as follows. We denote $A_i = d + a_i = A_\infty + \alpha_i$ and $A_\infty = d + a_\infty$. Firstly, given an L^p bound on the connection matrices, the bounds on $d_{A_\infty}^* \alpha_i$ and $d^* \alpha_i$ are equivalent. This is because, for any one-form $\alpha \in \Omega^1(\Omega, \mathfrak{g})$, $d_{A_\infty}^* \alpha = d^* \alpha + *[a_\infty \wedge * \alpha]$, and so,

$$\|d^* \alpha_i\|_{L^p} \leq c(\|\alpha_i\|_{L^p(\Omega)} + \|d_{A_\infty}^*(\alpha_i)\|_{L^p(\Omega)}).$$

Similarly the bounds on $\nabla_{A_\infty} \alpha_i$ and $\nabla \alpha_i$ are equivalent, so we proceed to bound α_i in $W^{1,p}$. Next, we write $F(A_i) = F(A_\infty) + d\alpha_i + [a_\infty \wedge \alpha_i] + [\alpha_i \wedge \alpha_i]$. For $1 \leq q \leq p$, on any compact domain in \mathbb{C} , there is a constant c such that

$$(33) \quad \|d\alpha_i\|_{L^q} \leq c(\|F(A_i) - F(A_\infty)\|_{L^p} + \|\alpha_i\|_{L^q} + \|\alpha_i\|_{L^{2q}}^2).$$

The doubling of the Sobolev exponent in the quadratic term above necessitates a bootstrapping procedure. First assume $2 < p < 4$. Set $q_0 := p$. There exists m and a sequence $2 < q_0 < \dots < q_{m-1} \leq 4$, $q_m > 4$ such that

$$q_0 = p, \quad q_i < \frac{2q_{i-1}}{4 - q_{i-1}} \quad i \geq 1.$$

We also fix a sequence of domains $\Omega = \Omega_0 \supset \dots \supset \Omega_{m+2} = \Omega'$ such that $\text{int}(\Omega_k) \supset \Omega_{k+1}$. Suppose $\alpha_i \in L^{q_i}(\Omega_i)$. Then, by Hölder's inequality $[\alpha_i \wedge \alpha_i] \in L^{q_i/2}(\Omega_i)$, and consequently by (33) $d\alpha_i \in L^{q_i/2}$. Now, we apply elliptic regularity to get $\alpha_i \in W^{1,q_i/2}(\Omega_{i+1})$ and finally, by Sobolev embedding $W^{1,q_i/2} \hookrightarrow L^{q_{i+1}}$. The estimates are as follows:

$$\begin{aligned} \|\alpha_i\|_{L^{q_{i+1}}(\Omega_{i+1})} &\leq c\|\alpha_i\|_{W^{1,q_i/2}(\Omega_{i+1})} \\ &\leq c(\|d\alpha_i\|_{L^{q_i/2}(\Omega_i)} + \|d^* \alpha_i\|_{L^{q_i/2}(\Omega_i)} + \|\alpha_i\|_{L^{q_i/2}(\Omega_i)}) \\ &\leq c(\|F(A_i) - F(A_\infty)\|_{L^{q_i/2}(\Omega_i)} + \|\alpha_i\|_{L^{q_i/2}(\Omega_i)} + \|\alpha_i\|_{L^{q_i}(\Omega_i)}^2). \end{aligned}$$

By repeating this procedure, we end up with $\alpha_i \in L^{q_m}(\Omega_m)$. If $p > 4$, then m is equal to 0 in the above discussion, and we straightaway have $\alpha_i \in L^{q_m}$. Applying elliptic regularity again, we get $\alpha_i \in W^{1,q_m/2}(\Omega') \hookrightarrow C^0(\Omega')$. With a C^0 bound on α_i , the bound on $d\alpha_i$ simplifies to

$$\|d\alpha_i\|_{L^p} \leq c(\|F(A_i) - F(A_\infty)\|_{L^p} + \|\alpha_i\|_{L^p} + \|\alpha_i\|_{L^p} \|\alpha_i\|_{L^\infty}).$$

Finally, by elliptic regularity $\alpha_i \in W^{1,p}(\Omega')$ and the norm is bounded by an expression that is a polynomial in the quantities $\|A_i - A_\infty\|_{L^p(\Omega)}$, $\|F(A_i) - F(A_\infty)\|_{L^p(\Omega)}$ and $\|d_{A_\infty}^*(A_i - A_\infty)\|_{L^p(\Omega)}$. If these quantities are small enough, the linear terms dominate in the polynomial and we get the estimate (32). \square

Proof. (Proof of Theorem 1.1) Given an affine vortex (A, u) , we will first construct a slice of the gauge group action at (A, u) and show that it is a Banach manifold. The space

$$\hat{\mathcal{B}}_{(A,u)}^{p,\delta} := \{(\alpha, \xi) \in W_{\text{loc}}^{1,p}(\Omega^1(\mathbb{C}, \mathfrak{g}) \times \Gamma(\mathbb{C}, u^*TX)) : \|(\alpha, \xi)\|_{p,\delta} < \infty\}$$

is a Banach manifold. The space $\mathcal{B}_{(A,u)}^{p,\delta}$ is cut out of $\hat{\mathcal{B}}_{(A,u)}^{p,\delta}$ as the zero of the function

$$\hat{\mathcal{B}}_{(A,u)}^{p,\delta} \ni (\alpha, \xi) \mapsto d_A^* \alpha + d\Phi(J\xi) \in L^{p,\delta}(\mathbb{C}, \mathfrak{g}).$$

This is a linear function, and so the kernel $\mathcal{B}_{(A,u)}^{p,\delta}$ is a Banach space, and hence a Banach manifold. In a neighborhood of (A, u) in $\mathcal{B}_{(A,u)}^{p,\delta}$, a neighborhood of the moduli space of affine vortices $M^G(\mathbb{C}, X)$ is cut out by the holomorphicity and vortex equations. To make this idea precise, we define an infinite-dimensional vector bundle $\mathcal{E}^{p,\delta} \rightarrow \mathcal{B}_{(A,u)}^{p,\delta}$, whose fiber over the point $(A', u') \in \mathcal{B}_{(A,u)}^{p,\delta}$ is given by

$$\mathcal{E}^{p,\delta}|_{A',u'} := L^{p,\delta}(\Omega^{0,1}(\mathbb{C}, (u')^*TX) + \Gamma(\mathbb{C}, \mathfrak{g})).$$

A neighborhood of $M^G(\mathbb{C}, X)$ centered at (A, u) is given by the zero set of the section

$$\mathcal{F} : \mathcal{B}_{(A,u)}^{p,\delta} \rightarrow \mathcal{E}^{p,\delta}, \quad (A', u') \mapsto (\bar{\partial}_{A'} u', *F_{A'} + \Phi(u')).$$

At the central point (A, u) , the derivative of \mathcal{F} is given by the first two terms of the augmented differential $\hat{\mathcal{D}}_{(A,u)}$. Since (A, u) is regular, $d\mathcal{F}$ is onto at (A, u) . Hence, there is a neighborhood of (A, u) in $\mathcal{B}_{(A,u)}^{p,\delta}$, denoted by $B_\epsilon(\mathcal{B}_{(A,u)}^{p,\delta})$, where $d\mathcal{F}$ is onto. By the implicit function theorem, the zero-level set of \mathcal{F} restricted to $B_\epsilon(\mathcal{B}_{(A,u)}^{p,\delta})$ is a Banach manifold. By Proposition 6.1, the kernel of this operator is of dimension $\dim(X) - 2\dim(G) + 2c_1(u^*TX)$. Therefore, the zero level set of \mathcal{F} in $B_\epsilon(\mathcal{B}_{(A,u)}^{p,\delta})$ is a manifold of the same dimension.

The transition functions between charts are smooth using the fact that gauge transforming to Coulomb gauge is a smooth operation by Remark 7.2. The fact that the manifold topology agrees with the compact convergence topology of vortices follows from Proposition 5.1. \square

6. FREDHOLM RESULT

In this section, we prove that the vortex differential operator at the vortex (A, u) is a Fredholm operator of the expected index. This result is proved by Ziltener [25], but we provide an alternate presentation.

We recall some ideas and terminology required to state the Fredholm result Proposition 6.1. The space

$$\hat{\mathcal{B}}_{(A,u)}^{p,\delta} := \{(\alpha, \xi) \in W_{\text{loc}}^{1,p}(\Omega^1(\mathbb{C}, \mathfrak{g}) \times \Gamma(\mathbb{C}, u^*TX)) : \|(\alpha, \xi)\|_{p,\delta} < \infty\}$$

was defined in the proof of Theorem 1.1. The vortex differential operator (3) extends to an operator between Sobolev completions

$$\begin{aligned} \hat{\mathcal{D}}_{(A,u)} : \hat{\mathcal{B}}_{(A,u)}^{p,\delta} &\rightarrow \Omega^{0,1}(\mathbb{C}, u^*TX)_{L^{p,\delta}} \times \Gamma(\mathbb{C}, \mathfrak{g} \oplus \mathfrak{g})_{L^{p,\delta}} \\ (\alpha, \xi) &\mapsto ((\nabla_A^X \xi + \alpha_X)^{0,1} - \frac{1}{2} J(\nabla_\xi^X J) \partial_{J,A} u, d\Phi(u)\xi + *d_A \alpha, d_A^* \alpha + d\Phi(J\xi)). \end{aligned}$$

The Fredholm index involves a Chern class term $c_1^G(\beta)$. We recall that for a bounded finite energy affine vortex, the singularity at infinity can be removed, and so the vortex (A, u) represents an equivariant homology class $\beta \in H_2^G(X)$, see Remark 2.5. The first equivariant Chern class $c_1^G = c_1^G(TX, J) \in H_G^2(X, \mathbb{Z})$ is defined as the ordinary first Chern class of the vector bundle $EG \times_G TX \rightarrow EG \times_G X$ equipped with the complex structure J . The pairing $c_1^G(\beta) := \langle c_1^G, \beta \rangle \in \mathbb{Z}$ is equal to the first Chern number of the complex vector bundle $u^*T^{\text{vert}}P(X)$ over \mathbb{P}^1 . On the trivialization $P|_{\mathbb{C}} \simeq \mathbb{C} \times G$, this bundle coincides with the pullback bundle u^*TX , and so while proving the proposition, we shorten notation and refer to $u^*T^{\text{vert}}P(X)$ as u^*TX .

Proposition 6.1. *Suppose (A, u) is an affine vortex with finite energy and bounded image, which represents the equivariant homology class $\beta \in H_2^G(X)$. The operator $\hat{\mathcal{D}}_{(A,u)}$ is Fredholm of index $\dim X - 2 \dim G + 2c_1^G(\beta)$.*

6.1. A weighted Sobolev norm for sections extendable to \mathbb{P}^1 . The first component of the vortex differential operator is a delbar operator on the space of sections of the pullback of the tangent bundle TX . By the removal of singularity for vortices, the pullback bundle extends to a bundle on \mathbb{P}^1 . In this Section, we introduce a weighted Sobolev norm on \mathbb{C} for sections of bundles on \mathbb{P}^1 . The definition of the norms uses a connection, but different choices of connections lead to equivalent norms. These ideas will be used to simplify the calculation of the Fredholm index of the vortex differential operator. The main result of this section is Lemma 6.3.

Remark 6.2. (A smooth structure on the pullback bundle u^*TX) The map $u : \mathbb{P}^1 \rightarrow P(X)$ is not smooth at infinity, it is only continuous. Therefore there is no canonical smooth structure on u^*TX . We can however fix a smooth structure via a trivialization of the tangent bundle TX in a neighborhood of $u(\infty)$. By the removal of singularity for vortices (Proposition 2.4) at infinity, after a gauge transformation on $\mathbb{C} \setminus B_R$, the connection A is of the form $d + a$ where $a \in W^{1,p,\delta}(\mathbb{C} \setminus B_R)$ and the map u extends continuously over infinity. Let $U \subset X$ be a neighborhood of $u(\infty)$. We fix a complex trivialization

$$(34) \quad U \times (\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}^{\bar{n}}) \rightarrow TX|_U,$$

where on any $x \in U$, $s \in \mathfrak{g}_{\mathbb{C}}$ is mapped to s_x and $\{x\} \times \mathbb{C}^{\bar{n}}$ is mapped to $(\mathfrak{g}_{\mathbb{C}})_x^{\perp}$.

The space of sections on a Hermitian vector bundle on \mathbb{P}^1 can be completed under a weighted Sobolev norm defined on \mathbb{C} . Suppose $E \rightarrow \mathbb{P}^1$ is a Hermitian vector bundle. Let $\mathcal{A}(E)$ denote the space of smooth unitary connections on E . Choose a connection $B \in \mathcal{A}(E)$. For any section $\sigma \in \Gamma(\mathbb{P}^1, E)$, define the norm

$$(35) \quad \|\sigma\|_{p,\delta} := \|\sigma\|_{L^\infty} + \|d_B \sigma\|_{L^{p,\delta}(\Omega^1(\mathbb{C}, E))}.$$

The Sobolev completion of sections under the above norm is denoted by $\Gamma(\mathbb{P}^1, E)_{p,\delta}$. In Lemma 6.3, we prove that the (p, δ) -norm is independent of the choice of connection B . On the vector bundle $u^*TX \rightarrow \mathbb{P}^1$, the covariant derivative ∇_A^X defined using the Levi-Civita connection on X is not smooth at infinity. It is also not complex linear. In the following lemma, we also show that in the vortex differential operator, ∇_A^X can be replaced by a smooth complex-linear Hermitian connection on \mathbb{P}^1 via a compact perturbation.

Lemma 6.3. (a) *The norm in (35) is independent of the choice of unitary connection.*

(b) *For any unitary connection B , the covariant derivative $\nabla_B : \Gamma(\mathbb{P}^1, E) \rightarrow \Omega^1(\mathbb{P}^1, E)$ extends to a bounded map between Sobolev completions*

$$\nabla_B : \Gamma(\mathbb{P}^1, E)_{p,\delta} \rightarrow L^{p,\delta}(\Omega^1(\mathbb{C}, E))$$

(c) *For two connections B_0, B_1 , the difference $\nabla_{B_0} - \nabla_{B_1}$ is compact.*

(d) *Suppose (A, u) is an affine vortex with finite energy and bounded image. Suppose u^*TX is equipped with a smooth structure as in Remark 6.2. For a smooth unitary connection B on the bundle $u^*TX \rightarrow \mathbb{P}^1$, the difference $\nabla_B - \nabla_A^X$ is compact and there is an equivalence of norms*

$$\|\sigma\|_{p,\delta} \sim \|\sigma\|_{L^\infty} + \|\nabla_A^X \sigma\|_{L^{p,\delta}(\mathbb{C}, u^*TX)}.$$

(e) *If E is a line bundle and $B \in \mathcal{A}(E)$ a unitary connection, the operator $\bar{\partial}_B : \Gamma(\mathbb{P}^1, E)_{p,\delta} \rightarrow L^{p,\delta}(\mathbb{C}, E)$ is Fredholm and has index $2c_1(E) + 2$.*

Proof. Given a section $\sigma \in \Gamma(\mathbb{P}^1, E)$ and unitary connections $B_0, B_1 \in \mathcal{A}(E)$, in a trivialization of E on $\mathbb{P}^1 \setminus B_R$, we can write $B_1 - B_0 = a \in \Omega^1(\mathbb{P}^1 \setminus B_R, \mathfrak{u}(n))$. With respect to the Fubini-Study metric on \mathbb{P}^1 , we have $a \in L^\infty(\mathbb{P}^1 \setminus B_1)$. This implies, in the Euclidean metric on $\mathbb{C} \setminus B_1$, $|a(z)| = O(|z|^{-2})$. Therefore on $\mathbb{C} \setminus B_1$, the difference is

$$\|\nabla_{B_0} \sigma - \nabla_{B_1} \sigma\|_{L^{p,\delta}} = \|[a, \sigma]\|_{L^{p,\delta}} \leq c \|\sigma\|_{L^\infty}.$$

We can then conclude the equivalence of the norms $\|\sigma\|_{L^\infty} + \|\mathrm{d}_{B_0} \sigma\|_{L^{p,\delta}(\mathbb{C})}$ and $\|\sigma\|_{L^\infty} + \|\mathrm{d}_{B_1} \sigma\|_{L^{p,\delta}(\mathbb{C})}$, which proves part (a). Part (b) is a straightforward consequence of part (a). The operator $\nabla_{B_0} - \nabla_{B_1}$ is compact by Proposition 2.10, since in the Euclidean metric, the term $|a|$ goes to zero as $z \rightarrow \infty$.

Part (d) is proved by showing that the contribution of the Levi-Civita connection matrices is a compact perturbation. The complex trivialization of the tangent bundle TX in the neighborhood U of $u(\infty)$ from (34) can be pulled back via the map u to produce a trivialization of $u^*TX|_{\mathbb{P}^1 \setminus B_R}$ for some $R > 0$. By scaling, we may assume the trivialization corresponds to an orthonormal frame of u^*TX . The Levi-Civita connection on U can be written as $\mathrm{d} + a^X$, where $a^X \in \Omega^1(U, \mathrm{End}_{\mathbb{R}}(TX))$. By part (a) of the Lemma, we can assume that the connection B on $u^*TX \rightarrow \mathbb{P}^1$ coincides with the trivial connection on $\mathbb{P}^1 \setminus B_R$. The difference between the covariant derivatives on $\mathbb{C} \setminus B_R$ is

$$\nabla_{A,v}^X \xi - \nabla_{B,v} \xi = a^X(\mathrm{d}u(v))\xi + a^X(\xi)a(v)_X, \quad z \in \mathbb{C}, v \in T_z \mathbb{C}, \xi : \mathbb{C} \setminus B_R \rightarrow u^*TX.$$

The operator $\xi \mapsto \nabla_{A,v}^X \xi - \nabla_{B,v} \xi$ is compact between the spaces $L^\infty(\mathbb{C} \setminus B_R) \rightarrow L^{p,\delta}$ for the following reason. The element a^X is in L^∞ and for any $\delta_1 \in (\delta, 1)$, $du(v)$, $a(v)_X$ are in W^{1,p,δ_1} , and the inclusion $W^{1,p,\delta_1} \hookrightarrow L^{p,\delta}$ is compact. The equivalence of norms can be proved in a similar way: on $\mathbb{C} \setminus B_R$,

$$\|\xi\|_{L^\infty} + \|\nabla_A^X \xi\|_{L^{p,\delta}} \sim \|\xi\|_{L^\infty} + \|\nabla_B \xi\|_{L^{p,\delta}}.$$

The proof of part (e) is by defining an equivalent norm using a local trivialization of the line bundle E . By parts (a) and (c) of the Lemma, it is enough to prove the result for a specific unitary connection B . So, we assume that B is flat in a neighborhood of infinity $\mathbb{P}^1 \setminus B_R$. The unitary connection B gives a holomorphic structure on E via the delbar operator $\bar{\partial}_B$. We choose holomorphic trivializations $E|_{\mathbb{C}}$ and $E|_{\mathbb{P}^1 \setminus \{0\}}$ that are related by the transition function $z \mapsto z^d$, where $d := c_1(E)$. Further, by the flatness of the connection near infinity, the trivialization $E|_{\mathbb{P}^1 \setminus \{0\}}$ can be chosen so that it is metric-preserving. Under the trivializations a section $\sigma \in \Gamma(\mathbb{P}^1, E)$ is described by $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ and $\hat{\sigma} : \mathbb{P}^1 \setminus B_R \rightarrow \mathbb{C}$. There is an equivalence of norms on $\mathbb{C} \setminus B_R$

$$\|\hat{\xi}\|_{L^\infty} + \|d\hat{\xi}\|_{L^{p,\delta}} \sim |\hat{\xi}(\infty)| + \|\hat{\xi} - \hat{\xi}(\infty)\|_{L^{p,\delta-1}} \sim |\hat{\xi}(\infty)| + \|\xi - \hat{\xi}(\infty)z^d\|_{L^{1,p,\delta-1-d}},$$

where the second equivalence is by Hardy's inequality (Lemma 2.9). Inside the ball, $W^{1,p}$ norms are equivalent for different choices of covariant derivatives or trivializations. Therefore, on \mathbb{C} there is an equivalence of norms

$$\|\hat{\xi}\|_{L^\infty(\mathbb{P}^1, E)} + \|d_A \xi\|_{L^{p,\delta}(\mathbb{C}, E)} \sim |\hat{\xi}(\infty)| + \|\xi - \hat{\xi}(\infty)z^d\|_{L^{1,p,\delta-1-d}(\mathbb{C}, \mathbb{C})},$$

In a similar way, we can prove that with the above trivialization, the norms on the spaces $L^{p,\delta}(\Omega^{0,1}(\mathbb{C}, E))$ and $L^{p,\delta-d}(\Omega^{0,1}(\mathbb{C}, \mathbb{C}))$ are equivalent. With these norm equivalences, we now have to prove that the operator

$$\bar{\partial}_B : \mathbb{C} \times L^{1,p,\delta-1-d} \rightarrow L^{p,\delta-d}, \quad (\sigma_\infty, \sigma) \mapsto \bar{\partial}_B(\sigma_\infty z^d + \sigma)$$

is Fredholm and has index $2d+2$. The operators $\bar{\partial}$ and $\bar{\partial}_B$ agree on $\mathbb{C} \setminus B_R$, therefore by a compact perturbation $\bar{\partial}_B$ can be replaced by $\bar{\partial}$. The result now follows from [23, Proposition 2.15]. \square

The following lemma related to Hermitian vector bundles on \mathbb{P}^1 is also used in the proof of the Fredholm result.

Lemma 6.4. *Given a Hermitian vector bundle $E \rightarrow \mathbb{P}^1$ and an orthonormal frame $\{s_i : \mathbb{P}^1 \setminus B_R \rightarrow E : 1 \leq i \leq n\}$ and d_1, \dots, d_n such that $\sum_i d_i = d := c_1(E)$. Then, there is a unitary connection B on E such that E splits as the sum of line bundles ℓ_1, \dots, ℓ_n satisfying $c_1(\ell_i) = d_i$ and ℓ_i is spanned by s_i in $\mathbb{P}^1 \setminus B_R$.*

Proof. Choose an orthonormal frame on the restriction of the bundle $E|_{B_{2R} \setminus B_R}$. The resulting transition function is given by $\phi : B_{2R} \setminus B_R \rightarrow U(n)$. Recall that $\pi_1(U(n)) = \mathbb{Z}$ and the winding number of the map ϕ is given by the first Chern class $d = c_1(E)$. We claim the trivialization inside the ball can be modified by a gauge transformation $g : B_{2R} \rightarrow U(n)$ so that the transition function is given by $\phi_1 : (r, \theta) \mapsto \text{diag}(e^{2\pi d_1 \theta/n}, \dots, e^{2\pi d_n \theta/n})$. The transition functions ϕ and ϕ_1 have the same winding number, and so the map $\phi_1 \circ \phi^{-1}$ is contractible and therefore

is extendable inside the ball to produce the required gauge transformation g . Now, the form of the transition function tells us that the bundle u^*TX splits into a sum of orthogonal line bundles ℓ_1, \dots, ℓ_n that satisfy the Lemma. \square

6.2. Proof of Fredholm result. The following is an observation used in the proof of the Fredholm result.

Remark 6.5. (An eigenvalue decomposition of $\mathfrak{g}_{\mathbb{C}}$) For any point $x \in \Phi^{-1}(0) \subset X$, denote the infinitesimal action of \mathfrak{g} by

$$L_x : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{g}_{\mathbb{C}} \quad s \mapsto s_X(x).$$

Under the metric $\omega(\cdot, J\cdot)$, the adjoint of this metric is $L_x^* : T_x X \rightarrow \mathfrak{g}_{\mathbb{C}}$ given by $v \mapsto d\Phi_x(Jv) + \iota d\Phi(v)$. Therefore the map $L_x^* L_x$ is symmetric and positive on $\mathfrak{g}_{\mathbb{C}}$ and has eigen-values $\lambda_1, \dots, \lambda_g > 0$. The corresponding eigen sections $\{s_1, \dots, s_g\}$ forms a complex orthogonal basis of $\mathfrak{g}_{\mathbb{C}}$. The tangent vectors $\{(s_1)_x, \dots, (s_g)_x\}$ are orthogonal in $T_x X$ with respect to the metric $\omega(\cdot, J\cdot)$ and form a complex basis of $(\mathfrak{g}_{\mathbb{C}})_x$ with respect to the Hermitian metric on $T_x X$.

Proof. (Proof of Proposition 6.1) We first choose a unitary connection on the bundle u^*TX in a way that the bundle splits into a direct sum of line bundles in a convenient way. We fix some notation

$$g := \dim G, \quad n := \frac{1}{2} \dim X, \quad \bar{n} := n - 2g = \frac{1}{2} \dim X // G, \quad d := c_1^G(\beta).$$

We carry out the proof assuming $\bar{n} > 0$. The other case of $\bar{n} = 0$ is discussed in the end. The bundle $u^*TX \rightarrow \mathbb{P}^1$ is a Hermitian complex vector bundle. Firstly, we fix an orthonormal frame $\{s_1, \dots, s_n\}$ of $u^*TX|_{\mathbb{P}^1 \setminus B_R}$ such that the following are satisfied:

- (a) The first g sections s_1, \dots, s_g span the sub-bundle $(\mathfrak{g}_{\mathbb{C}})_u|_{\mathbb{P}^1 \setminus B_R}$, and hence the other sections s_{g+1}, \dots, s_n span the complement $(\mathfrak{g}_{\mathbb{C}})_u^\perp|_{\mathbb{P}^1 \setminus B_R}$.
- (b) At infinity, the frame is given by eigen-vectors of the map $L_{u(\infty)} L_{u(\infty)}^*$ and the corresponding eigen-values are $\lambda_1, \dots, \lambda_g$ respectively. See Remark 6.5.

Next, by Lemma 6.4, there is a unitary connection B on u^*TX , such that the bundle splits as a direct sum of orthogonal line bundles ℓ_1, \dots, ℓ_n of degree $0, \dots, 0, d$ and such that $\ell_i|_{\mathbb{P}^1 \setminus B_R} = \text{span}(s_i)$. By Lemma 6.3, there is an equivalence of norms

$$\begin{aligned} \|\xi\|_{L^\infty} + \|\nabla_A^X \xi\|_{L^{p,\delta}} + \|d\Phi(\xi)\|_{L^{p,\delta}} + \|d\Phi(J\xi)\|_{L^{p,\delta}} \\ \sim \|\xi\|_{L^\infty} + \|\nabla_B \xi\|_{L^{p,\delta}} + \|\pi_{\mathfrak{g}_{\mathbb{C}}} \xi\|_{L^{p,\delta}(\mathbb{C} \setminus B_R)}, \end{aligned}$$

and in the differential operator $(\nabla_A^X)^{0,1}$ can be replaced by delbar operator $\bar{\partial}_B$ by a compact perturbation. The connection B can be assumed to be a flat connection on the trivial line bundles $\ell_1, \dots, \ell_{n-1}$. At infinity, there is an identification of $\mathfrak{g}_{\mathbb{C}}$ to the fiber of the sub-bundle $\sum_{i=1}^g \ell_i|_\infty$, where $s \in \mathfrak{g}$ is identified to $s_{u_\infty} \in T_{u(\infty)} X$. Since each of these line bundles is a trivial bundle with trivial connection on \mathbb{P}^1 , the identification at ∞ extends to a holomorphic map

$$(36) \quad \mathbb{P}^1 \times \mathfrak{g}_{\mathbb{C}} \rightarrow \oplus_{i=1}^g \ell_i =: \ell_{\mathfrak{g}_{\mathbb{C}}}.$$

The various operators in $\mathcal{D}_{(A,u)}$ can be modified in the following ways by compact perturbations. The space $\Omega^1(\mathbb{C}, \mathfrak{g})$ is identified with $\Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}})$ by sending

$$(37) \quad \Omega^1(\mathbb{C}, \mathfrak{g}) \ni \alpha_s ds + \alpha_t dt \rightarrow \alpha_s + \iota \alpha_t \in \Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}).$$

The space $\Omega^{0,1}(\mathbb{C}, \ell_{\mathfrak{g}_{\mathbb{C}}})$ is same as $\Gamma(\mathbb{C}, \ell_{\mathfrak{g}_{\mathbb{C}}}) \otimes d\bar{z}$, which can be identified to $\Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}})$ by (36).

- (a) $((\nabla_A^X \xi)^{0,1})$ By Lemma 6.3 (d), this term can be replaced by $\bar{\partial}_B$.
- (b) $(\frac{1}{2} J(\tilde{\nabla}_\xi^X J) \partial_{J,A} u)$ This term can be dropped as it is compact, because $|\partial_{J,A} u| \rightarrow 0$ as $z \rightarrow \infty$.
- (c) $(\alpha_X^{0,1})$ On $\mathbb{C} \setminus B_R$, the image of this term is contained in the sub-bundle $\Omega^{0,1}(\mathbb{C}, \ell_{\mathfrak{g}_{\mathbb{C}}})$, therefore by a compact perturbation the contributions to the complementary subbundle $\Omega^{0,1}(\mathbb{C}, \oplus_{i=g+1}^n \ell_i)$ can be dropped. Via the choice of the trivialization (36) of the bundle $\ell_{\mathfrak{g}_{\mathbb{C}}}$, the fiberwise map

$$\Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}) \simeq \Omega^1(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}) \xrightarrow{\alpha_X} \Omega^{0,1}(\mathbb{C}, \ell_{\mathfrak{g}_{\mathbb{C}}}) \simeq \Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}).$$

approaches identity as $z \rightarrow \infty$. Therefore, by a compact perturbation α_X can be replaced by $\text{Id} : \Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}) \simeq \Gamma(\mathbb{C}, \ell_{\mathfrak{g}_{\mathbb{C}}})$.

- (d) $(*d_A, d_A^*)$ The terms $d_A \alpha$ and $d_A^* \alpha$ can be replaced by $d\alpha$ and $d^* \alpha$ respectively by a compact perturbation. On $\mathbb{C} \setminus B_R$, the connection A is of the form $A = d + \eta d\theta + a$, where $a \in W^{1,p,\delta} \hookrightarrow C^{0,\delta-1-\frac{2}{p}}$ and the twist term has size $|\eta d\theta(z)| = O(|z|^{-1})$. Both these terms decay to zero as $z \rightarrow \infty$. Therefore on $\mathbb{C} \setminus B_R$, the difference

$$d_A \alpha - d\alpha = [\eta d\theta \wedge \alpha] + [a \wedge \alpha] : W^{1,p,\delta} \rightarrow L^{p,\delta}$$

is a compact perturbation. Further, under the identification (37), we have $*d \oplus \iota d^* = \frac{\partial}{\partial z} : \Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}) \rightarrow \Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}})$.

- (e) $(d\Phi(J\xi), d\Phi(\xi))$ Outside a ball B_R , these operators are zero on the sub-bundle $\oplus_{i=g+1}^n \ell_i$. When restricted to the sub-bundle $\ell_{\mathfrak{g}_{\mathbb{C}}}$ (which is identified to $\mathbb{C} \times \mathfrak{g}_{\mathbb{C}}$ by (36)), as $z \rightarrow \infty$ the operator approaches the limit

$$\Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}) \ni s \mapsto d\Phi(s_{u(\infty)}) + \iota d\Phi(Js_{u(\infty)}) \in \Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}),$$

which is just multiplication by the constant diagonal matrix $\text{diag}(\lambda_1, \dots, \lambda_g)$ (see Remark 6.5). Therefore, by a compact perturbation $d\Phi(J\cdot) + \iota d\Phi(\cdot)$ can be replaced by the constant diagonal matrix.

Using the above discussion, we can now re-write the differential operator and the domain and target spaces. The domain space is

$$\begin{aligned} \mathcal{B}_{(A,u)}^{p,\delta} &= \Omega^1(\mathbb{C}, \mathfrak{g})_{W^{1,p,\delta}} + \oplus_{i=1}^g (\Gamma(\mathbb{C}, \ell_i)_{p,\delta} \cap \Gamma(\mathbb{C}, \ell_i)_{L^{p,\delta}}) + \oplus_{i=g+1}^n \Gamma(\mathbb{C}, \ell_i)_{p,\delta} \\ &= W^{1,p,\delta}(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}) + W^{1,p,\delta}(\mathbb{C}, \mathfrak{g}_{\mathbb{C}}) + \oplus_{i=g+1}^n \Gamma(\mathbb{C}, \ell_i)_{p,\delta}. \end{aligned}$$

In the last line the three terms are the connection term, map term parallel to $\mathfrak{g}_{\mathbb{C}}$ -action and the map term perpendicular to $\mathfrak{g}_{\mathbb{C}}$ -action respectively. The target space

can be rewritten as

$$\begin{aligned} & \Omega^{0,1}(\mathbb{C}, \oplus_{i=1}^n \ell_i)_{L^{p,\delta}} + \Gamma(\mathbb{C}, \mathfrak{g} \oplus \mathfrak{g})_{L^{p,\delta}} \simeq \\ & \Omega^{0,1}(\mathbb{C}, \mathfrak{g}_{\mathbb{C}})_{L^{p,\delta}} + (\oplus_{i=g+1}^n \Omega^{0,1}(\mathbb{C}, \ell_i)_{L^{p,\delta}}) + \Gamma(\mathbb{C}, \mathfrak{g}_{\mathbb{C}})_{L^{p,\delta}}. \end{aligned}$$

In the last line, the first two spaces are the target space for the differential of the delbar operator; the first one is parallel to the $\mathfrak{g}_{\mathbb{C}}$ orbit and the second one is perpendicular. The third space is for the sum of the differential of the vortex equation and the vortex Coulomb gauge operator. The differential of the map terms that are perpendicular to the $\mathfrak{g}_{\mathbb{C}}$ -action can be written down as a direct sum of operators

$$(38) \quad \bar{\partial}_{\ell_i} : \Gamma(\mathbb{C}, \ell_i)_{p,\delta} \rightarrow L^{p,\delta}(\Omega^1(\mathbb{C}, \ell_i)), \quad i = g+1, \dots, n.$$

The sum of these operators is Fredholm and has index $2\bar{n} + 2d$, by Lemma 6.3 (e). The differential of the map term parallel to the $\mathfrak{g}_{\mathbb{C}}$ orbit is combined with the differential of the vortex equation and the Coulomb gauge operator, and can be expressed as

$$(39) \quad (\alpha, \pi_{\ell_{\mathfrak{g}_{\mathbb{C}}}} \xi) \mapsto (\bar{\partial}(\pi_{\ell_{\mathfrak{g}_{\mathbb{C}}}} \xi) + \alpha_X^{0,1}, *d\alpha + \iota d^* \alpha + \lambda \cdot (\pi_{\ell_{\mathfrak{g}_{\mathbb{C}}}} \xi)),$$

where in the last term we take dot product with the tuple $\lambda := (\lambda_1, \dots, \lambda_g)$, which are the eigenvalues of the operator $s \mapsto d\Phi(Js_{u_\infty})$ (see Remark 6.5). Decomposing $\mathfrak{g}_{\mathbb{C}}$ into an eigen-basis of this operator, and applying a compact perturbation to the operator as discussed earlier in the proof, the differential operator in (39) splits into a direct sum

$$(40) \quad \begin{pmatrix} \text{Id} & \frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial \bar{z}} & \lambda_i \end{pmatrix} : W^{1,p,\delta}(\mathbb{C}, \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}) \rightarrow L^p(\mathbb{C}, \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}).$$

Ziltener ([23, Proposition 2.18]) proves that this operator is Fredholm of index 0, by relating it to Calderon's operator $\text{Id} + \Delta : W^{k+1,p} \rightarrow W^{k-1,p}$ which is an isomorphism.

Finally, we consider the case when the symplectic quotient is trivial, i.e. $\bar{n} = 0$. In the above proof, the bundle u^*TX is split in a way that the component $(\mathfrak{g}_{\mathbb{C}})_u|_{\mathbb{P}^1 \setminus B_R}$ is part of a trivial vector bundle. Roughly speaking, all the twist in u^*TX is absorbed by components that are perpendicular to $(\mathfrak{g}_{\mathbb{C}})_u$ near infinity. This strategy fails when $\bar{n} = 0$, because in the neighborhood of infinity u^*TX coincides with $(\mathfrak{g}_{\mathbb{C}})_u$. To get around this issue, we artificially increase the dimension of X . We define $\hat{X} := X \times \mathbb{C}$ and let G act trivially on the second component. Then, the moment map Φ lifts to a moment map on \hat{X} . We view the vortex (A, u) as mapping to $X \times \{0\}$. The augmented differential operator with respect to the spaces X, \hat{X} are denoted by $\hat{D}_{(A,u)}^X$ and $\hat{D}_{(A,u)}^{\hat{X}}$ respectively. The domain and target space of $\hat{D}_{(A,u)}^{\hat{X}}$ has an extra summand $\Gamma(\mathbb{C}, \mathbb{C})_{p,\delta}$ and $L^{p,\delta}(\Omega^1(\mathbb{C}, \mathbb{C}))$ respectively. The differential operator $\hat{D}_{(A,u)}^{\hat{X}}$ splits diagonally as $\hat{D}_{(A,u)}^X + \bar{\partial}$, where $\bar{\partial}$ is defined between the spaces $\Gamma(\mathbb{C}, \mathbb{C})_{p,\delta} \rightarrow L^{p,\delta}(\Omega^{0,1}(\mathbb{C}, \mathbb{C}))$. This operator is onto and has a two dimensional kernel consisting of complex-valued constant functions (see Lemma 6.3 (e)). Now, we can apply the result for the case $\bar{n} > 0$ on \hat{X} to get the result on X . \square

7. COULOMB GAUGE FOR AFFINE VORTICES

A Euclidean chart for the moduli space of affine vortices centered at (A, u) consists of nearby vortices that are in Coulomb gauge with respect to (A, u) . This chart is a slice of the gauge group action at (A, u) . This idea is inspired by the Coulomb gauge condition for connections, which is the special case where the target space X is trivial (see Section 4.2.1 in Donaldson-Kronheimer [6]). In this section, we show that gauged maps close to (A, u) can be gauge transformed using small gauge transformations so that they are in Coulomb gauge with respect to (A, u) .

The statement of the results of this section require weighted Sobolev spaces with negative indices. These are defined by dualizing under the L^2 -pairing. For $k \in \mathbb{Z}_{\geq 0}$, $1 < p < \infty$ and $\delta \in \mathbb{R}$, and a compactly supported smooth function $\sigma \in C_0^\infty(\mathbb{C})$, we define a norm

$$\|\sigma\|_{W^{-k,p,\delta}} := \sup\left\{\int_{\mathbb{C}} f\sigma : f \in W^{k,p',-\delta}, \|f\|_{W^{k,p',-\delta}} = 1\right\},$$

where $p' := \frac{p}{p-1}$. The space $W^{-k,p}(\mathbb{C})$ is the completion of $C^\infty(\mathbb{C})$ under the norm $\|\cdot\|_{W^{-k,p,\delta}}$. It is the dual of $W^{-k,p',-\delta}(\mathbb{C})$. By dualizing the continuous map $\nabla : W^{k,p,\delta} \rightarrow W^{k-1,p,\delta}$ for $k \geq 1$, we see that it is also continuous for the negative index space. That is, $\nabla : W^{-k,p,\delta} \rightarrow W^{-k-1,p,\delta}$ is continuous for $k \geq 0$. Now, we can state the main result of the section.

Proposition 7.1. (Transforming to Coulomb gauge) *Suppose $k = 0, 1$, $p > 2$ and $\delta > 1 - \frac{2}{p}$. Suppose (A, u) is an affine vortex with finite energy and bounded image. Suppose $A = d + \eta d\theta + a$, where $\eta \in \frac{1}{2\pi} \exp^{-1}(\text{Id})$ and $a \in W^{1,p,\delta}$. There are constants c_1, c_2 so that the following holds. For any deformation $(\alpha, \xi) \in \Omega^1(\mathbb{C}, \mathfrak{g})_{W^{k,p,\delta}} \times \Gamma(\mathbb{C}, u^*TX)_{L^\infty}$ satisfying*

$$\|\alpha\|_{W^{k,p,\delta}} + \|\xi\|_{L^\infty} + \|d_A^* \alpha + d\Phi_u(J\xi)\|_{W^{k-1,p,\delta}} < c_1,$$

there is a gauge transformation e^s satisfying $\|s\|_{W^{k+1,p,\delta}} \leq c_2 \|d_A^ \alpha + d\Phi_u(J\xi)\|_{W^{k-1,p,\delta}}$ and such that $e^s(A + \alpha, \exp_u \xi)$ is in Coulomb gauge with respect to (A, u) .*

Proof. (Proof of Proposition 7.1) The proof is by implicit function theorem on the operator defined as

$$(41) \quad \begin{aligned} \mathcal{T}^{(\alpha,\xi)} : B_\epsilon(W^{k+1,p,\delta}(\mathbb{C}, \mathfrak{g})) &\rightarrow W^{k-1,p,\delta}(\mathbb{C}, \mathfrak{g}) \\ s &\mapsto d_A^*(e^s(A + \alpha) - A) + d\Phi_u(J \exp_u^{-1}(e^s \exp_u \xi)). \end{aligned}$$

We denote by inj_u the infimum of the injectivity radius for points on the image of u . We choose c_1 in the statement of the Proposition and ϵ in a way that $\|d_X(u, e^s \exp_u \xi)\|_{L^\infty}$ is smaller than inj_u , so that \exp^{-1} is well-defined in the definition of \mathcal{T} in (41) above. Before we proceed further, we define a function, which is useful for compact notation. For $x \in X$, define

$$R_x : T_x X \rightarrow \text{End}(\mathfrak{g}), \quad \xi \mapsto (\mathfrak{g} \ni s \mapsto d\Phi_x(J\Psi_x(\xi)^{-1}(s_{\exp_x \xi}))),$$

where $\Psi_u(\xi) : u^*TX \rightarrow (\exp_u \xi)^*TX$ denotes parallel transport along the geodesic $\exp_u t\xi$ with respect to the Levi-Civita connection. The map R_x is smooth and

varies smoothly with x . It induces a bundle map

$$R_u : \Gamma(\mathbb{C}, u^*TX) \rightarrow \Gamma(\mathbb{C}, \text{End}(\mathfrak{g})), \quad R_u(\xi)(z) := R_{u(z)}(\xi(z)) \forall z \in \mathbb{C}.$$

The linearization of \mathcal{T} at $s \in W^{k+1,p,\delta}(\mathbb{C}, \mathfrak{g})$ is given by

$$\begin{aligned} D\mathcal{T}_s^{(\alpha,\xi)} : W^{k+1,p,\delta}(\mathbb{C}, \mathfrak{g}) &\rightarrow W^{k-1,p,\delta}(\mathbb{C}, \mathfrak{g}), \\ s_1 &\mapsto d_A^* d_{e^s(A+\alpha)} s_1 + R_u(\xi_s) s_1 \end{aligned}$$

where $\xi_s := \exp_u^{-1}(e^s \exp_u \xi)$. The expression $D\mathcal{T}_s^{(\alpha,\xi)}(s_1)$ is continuous in the parameters (s, s_1) , and so $\mathcal{T}^{(\alpha,\xi)}$ is differentiable.

STEP 1: *There is a constant c_1 such that if $\|\alpha\|_{W^{k,p,\delta}} + \|\xi\|_{L^\infty} < c_1$, then $D\mathcal{T}_0^{(\alpha,\xi)}$ is invertible and the inverse $Q^{(\alpha,\xi)}$ is uniformly bounded. In particular, $\|Q^{(\alpha,\xi)}\| \leq C$.* By Proposition 7.3 below, the operator $D\mathcal{T}_0^0$ is invertible. With a suitable bound on (α, ξ) , we will show that $D\mathcal{T}_0^{(\alpha,\xi)} - D\mathcal{T}_0^0$ is small. The difference between the differentials is

$$(D\mathcal{T}_0^{(\alpha,\xi)} - D\mathcal{T}_0^0)s = d_A^*[\alpha, s] + (R_u(\xi) - R_u(0))s$$

Further, since the image of u is contained in a compact set, there exist constants c_1, c such that for all $x \in \text{im}(u)$ and $\xi \in B_{c_1}(T_x X)$, $|R_x(\xi)| \leq c|\xi|$. Hence, the bundle map R_u satisfies $\|R_u \xi\|_{L^\infty} \leq c\|\xi\|_{L^\infty}$ if $\|\xi\|_{L^\infty} < c_1$. Therefore,

$$\|(D\mathcal{T}_0^{(\alpha,\xi)} - D\mathcal{T}_0^0)s\|_{W^{k-1,p,\delta}} \leq c(\|\alpha\|_{W^{k,p,\delta}} + \|\xi\|_{L^\infty})\|s\|_{W^{k+1,p,\delta}}.$$

The difference $\|D\mathcal{T}_0^{(\alpha,\xi)} - D\mathcal{T}_0^0\|$ can be made smaller than $\frac{1}{2}\|D\mathcal{T}_0(0)^{-1}\|$ by choosing c_1 sufficiently small. In that case $D\mathcal{T}^{(\alpha,\xi)}(0)$ is invertible and $\|D\mathcal{T}^{(\alpha,\xi)}(0)^{-1}\| \leq 2\|D\mathcal{T}_0(0)^{-1}\|$.

STEP 2: *There exists constants c_1, c such that if the deformation (α, ξ) satisfies $\|\alpha\|_{W^{k,p,\delta}} + \|\xi\|_{L^\infty} < c_1$, then, for $s \in B_\epsilon(W^{k+1,p,\delta}(\mathbb{C}, \mathfrak{g}))$,*

$$\|D\mathcal{T}_s^{(\alpha,\xi)} - D\mathcal{T}_0^{(\alpha,\xi)}\| \leq c\|s\|_{W^{k+1,p,\delta}}.$$

We need to bound the expression

$$(D\mathcal{T}_s^{(\alpha,\xi)} - D\mathcal{T}_0^{(\alpha,\xi)})s_1 = d_A^*[(e^s(A+\alpha) - (A+\alpha)), s_1] + (R_u(\xi_s) - R_u(\xi))s_1,$$

Since the norms of α and s can be chosen to be small, there is a constant c such that

$$\|e^s(A+\alpha) - (A+\alpha)\|_{W^{k,p,\delta}} \leq c\|s\|_{W^{k+1,p,\delta}}, \quad \|R_u(\xi_s) - R_u(\xi)\|_{L^\infty} \leq \|s\|_{L^\infty}.$$

Therefore,

$$\|(D\mathcal{T}_s^{(\alpha,\xi)} - D\mathcal{T}_0^{(\alpha,\xi)})s_1\|_{W^{k-1,p,\delta}} \leq c\|s\|_{W^{k+1,p,\delta}}\|s_1\|_{W^{k+1,p,\delta}}.$$

STEP 3: *Finishing the proof.*

By the result in Step 2, there is a constant δ_0 such that

$$\|s\|_{W^{k+1,p,\delta}} < \delta_0 \implies \|D\mathcal{T}_s^{(\alpha,\xi)} - D\mathcal{T}_0^{(\alpha,\xi)}\| \leq \frac{1}{2C}.$$

The constant c_1 can be chosen so that the conditions in Step 1 and 2 are satisfied and in addition $\|\mathcal{T}^{(\alpha,\xi)}(0)\| = \|d_A^* \alpha + d\Phi(J\xi)\|_{W^{k-1,p,\delta}} < \frac{\delta_0}{8C}$. Finally, apply Proposition 7.7 with $\delta := 8C\|\mathcal{T}^{(\alpha,\xi)}(0)\|$. This ensures $\|\mathcal{T}^{(\alpha,\xi)}(0)\| < \frac{\delta}{4C}$ and we get a solution $s \in B_\delta(W^{1,p,\delta})$ for $\mathcal{F}^{(\alpha,\xi)}(s) = 0$. The solution s also satisfies $\|s\|_{W^{k+1,p,\delta}} < 8C\|\mathcal{T}^{(\alpha,\xi)}(0)\|$. Proposition 7.1 is proved by taking $c_2 = 8C$. \square

Remark 7.2. (Transforming to Coulomb gauge is a smooth operation) In a neighborhood of an affine vortex (A, u) given by $\hat{\mathcal{B}} := \{(A + \alpha, \exp_u \xi) : \|(\alpha, \xi)\|_{\delta,p} < c_1\}$, Proposition 7.1 induces a function $s : \hat{\mathcal{B}} \rightarrow L^{p,\delta}(\mathbb{C}, \mathfrak{g})$ such that $e^{s(\alpha,\xi)}(A + \alpha, \exp_u \xi)$ is in Coulomb gauge with respect to (A, u) . We claim that s is smooth. This can be seen by applying implicit function theorem on the function

$$\mathcal{T} : \hat{\mathcal{B}} \times W^{2,p,\delta}(\mathbb{C}, \mathfrak{g}) \rightarrow L^{p,\delta}(\mathbb{C}, \mathfrak{g}), \quad ((\alpha, \xi), \zeta) \mapsto d_A^*(e^\zeta(A + \alpha) + d\Phi(J \exp_u^{-1}(e^\zeta \exp_u \xi))),$$

which is a smooth map wherever it is defined. The partial derivative $\partial_\zeta \mathcal{T}$ is onto in a neighborhood of $(0, 0)$, and therefore s is smooth.

We remark that Proposition 7.1 was proved under the assumption that the central vortex (A, u) is in a certain gauge, where the connection matrices are asymptotically bounded. This condition is not significant because the vortex Coulomb gauge is gauge-invariant, i.e. a pair (A', u') being in Coulomb gauge with respect to (A, u) is equivalent to $g(A, u)$ being in Coulomb gauge with respect to $g(A', u')$ where g is a gauge transformation.

In the rest of this Section we prove the invertibility of the operator $\zeta \mapsto d_A^* d_A \zeta + d\Phi(J\zeta_u)$, where (A, u) is a vortex. The implicit function theorem argument in the proof of Proposition 7.1 was based on this result.

Proposition 7.3. *Let $k = 0, 1$, $p > 2$ and $1 - \frac{2}{p} < \delta < 1$. Suppose (A, u) is a finite energy affine vortex with bounded image, and that for some $R > 0$, $A = d + \eta d\theta + a$ on $\mathbb{C} \setminus B_R$, where $\eta \in \frac{1}{2\pi} \exp^{-1}(\text{Id}) \subset \mathfrak{g}$ and $a \in W^{1,p,\delta}(\Omega^1(\mathbb{C} \setminus B_R, \mathfrak{g}))$. Then, the operator*

$$(42) \quad W^{k+1,p,\delta}(\mathbb{C}, \mathfrak{g}) \rightarrow W^{k-1,p,\delta}(\mathbb{C}, \mathfrak{g}), \quad \zeta \mapsto d_A^* d_A \zeta + d\Phi_u(J\zeta_u)$$

is an isomorphism.

Proposition 7.3 is proved using the following Lemma.

Lemma 7.4. *Suppose $k = 0$ or 1 , A is a connection on the trivial bundle and $u : \mathbb{C} \rightarrow X$ is a map satisfying the following: there exist $R > 0$, $\eta \in \frac{1}{2\pi} \exp^{-1}(\text{Id})$ and $x \in \Phi^{-1}(0)$ such that on $\mathbb{C} \setminus B_R$, $A = d + \eta d\theta$ and $u(r, \theta) = e^{\eta\theta} x$. Then, the operator*

$$(43) \quad \mathcal{L} : W^{k+1,p,\delta}(\mathbb{C}, V) \rightarrow W^{k-1,p,\delta}(\mathbb{C}, V), \quad \zeta \mapsto d_A^* d_A \zeta + u^* d\Phi(J\zeta_u)$$

is an isomorphism for all $p > 1$ and $\delta \in \mathbb{R}$.

The proof of Lemma 7.4 is based on Calderon's result.

Lemma 7.5. (Calderon's theorem, [14, Theorem V.3]) *The operator*

$$\text{Id} + d^* d : W^{k+1,p}(\mathbb{C}, \mathbb{R}) \rightarrow W^{k-1,p}(\mathbb{C}, \mathbb{R})$$

is an isomorphism for all $p > 1$.

Proof of Lemma 7.4. The proof is first carried out for $\delta = 0$. We start by showing that on the complement of the ball B_R , the operator \mathcal{L} is a direct sum of operators of the form $\text{Id} + \Delta$. Observe that the operator $L_x := \zeta \mapsto d\Phi_x(\zeta_x) \in \text{End}(\mathfrak{g})$ is diagonalizable and has positive eigen-values $\lambda_1, \dots, \lambda_n$. Further, on $\mathbb{C} \setminus B_R$, $e^{\eta\theta} \mathcal{L} e^{-\eta\theta} = d^*d + L_x$. Therefore, the bundle $(\mathbb{C} \setminus B_R) \times \mathfrak{g}$ splits into line bundles $\ell_i := (\mathbb{C} \setminus B_R) \times \mathbb{C}$ for $1 \leq i \leq n$ on which $d^*d + L_x|_{\ell_i} = d^*d + \lambda_i \text{Id}$.

A twisted version of the operator \mathcal{L} is invertible; we use this fact to construct an inverse of \mathcal{L} outside the ball B_R . On \mathbb{C} , the operator $d^*d + \lambda_i : W^{k+1,p}(\mathbb{C}, \mathbb{R}) \rightarrow W^{k-1,p}(\mathbb{C}, \mathbb{R})$ is an isomorphism using Lemma 7.5 below and a scaling argument. Therefore, there is an inverse $\mathcal{K} : W^{k-1,p}(\mathbb{C}, \mathfrak{g}) \rightarrow W^{k+1,p}(\mathbb{C}, \mathfrak{g})$ of the operator

$$d^*d + L_x : W^{k+1,p}(\mathbb{C}, \mathfrak{g}) \rightarrow W^{k-1,p}(\mathbb{C}, \mathfrak{g}).$$

We will now construct an inverse outside the ball B_{2R} using \mathcal{K} . Consider $f \in W^{k-1,p}(\mathbb{C}, \mathfrak{g})$. Choose a radially symmetric cut-off function $\beta : \mathbb{C} \rightarrow [0, 1]$ that is 1 in B_R and 0 in $\mathbb{C} \setminus B_{2R}$. Define $\hat{f} \in W^{k-1,p}(\mathbb{C}, \mathfrak{g})$ as being equal to $\text{Ad}_{e^{-\eta\theta}} f$ on $\mathbb{C} \setminus B_R$ and zero on B_R . By the previous paragraph, $\hat{u} := \mathcal{K}\hat{f} \in W^{k+1,p}(\mathbb{C})$ is well-defined. Define $u_{\text{out}} := \text{Ad}_{e^{\eta\theta}}(\beta\hat{u})$. By our construction, we have

$$\|u_{\text{out}}\|_{W^{k+1,p}(\mathbb{C})} \leq c\|f\|_{W^{k-1,p}}, \quad \mathcal{L}u_{\text{out}} = f \text{ on } \mathbb{C} \setminus B_R.$$

Now, we extend the inverse to inside the ball B_R . We need to find a section $u_{\text{in}} \in W^{k+1,p}(B_{2R}, \mathfrak{g})$ satisfying

$$\mathcal{L}u_{\text{in}} = f - \mathcal{L}u_{\text{out}} \text{ on } B_{2R}, \quad u|_{\partial B_{2R}} = 0.$$

The above equation is an elliptic pde with Dirichlet boundary condition, and therefore has a unique solution u_{in} . We extend u_{in} by zero outside the ball B_{2R} and set $u := u_{\text{in}} + u_{\text{out}}$. The section u is in $W^{1,p}$ and weakly satisfies $\mathcal{L}u = f$. In case $k = 1$, by an elliptic regularity argument, we can show that u is in $W^{2,p}$. By construction, the solution u satisfies the norm bound $\|u\|_{W^{k+1,p}(\mathbb{C})} \leq c\|f\|_{W^{k-1,p}}$.

Next, we show that \mathcal{L} is injective using the self-adjointness of the operator. Suppose $u \in W^{k+1,p}(\mathbb{C}, \mathfrak{g})$ and $\mathcal{L}u = 0$. Then, $\int_{\mathbb{C}} \langle \mathcal{L}u, f \rangle = 0$ for all $f \in W^{k+1,p'}$ where $p' = \frac{p}{p-1}$. Integrating by parts, we see that u satisfies $\int_{\mathbb{C}} \langle u, \mathcal{L}f \rangle = 0$ for all $f \in W^{k+1,p'}$. But, we know that $\mathcal{L} : W^{k+1,p'} \rightarrow W^{k-1,p'}$ is onto. Therefore, $u = 0$.

Finally, we extend the result to non-zero values of δ . The map $W^{k,p} \ni u \mapsto (1 + |z|^2)^{-\delta/2} u \in W^{k,p,\delta}$ is an isomorphism for $k = 0, 2$. The map \mathcal{L}_δ defined as the composition

$$W^{k+1,p,\delta} \xrightarrow{(1+|z|^2)^{\delta/2}} W^{k+1,p} \xrightarrow{\mathcal{L}} W^{k-1,p} \xrightarrow{(1+|z|^2)^{-\delta/2}} W^{k-1,p,\delta},$$

is an isomorphism, and it differs from \mathcal{L} by a compact perturbation (see Proposition 2.10). Therefore, the map $\mathcal{L} : W^{k+1,p,\delta} \rightarrow W^{k-1,p,\delta}$ is a Fredholm operator with index zero. The kernel of this map is a subspace of the kernel of the map $\mathcal{L} : W^{k+1,p} \rightarrow W^{k-1,p}$, because $W^{k+1,p,\delta} \subset W^{k+1,p}$, and therefore the kernel is trivial. This proves that the map $\mathcal{L} : W^{k+1,p,\delta} \rightarrow W^{k-1,p,\delta}$ is an isomorphism. \square

Proof of Proposition 7.3. We prove the Proposition by showing that the operator (42) can be modified via a compact perturbation to an operator of the form (43)

in Lemma 7.4. By the removal of singularities (Proposition 2.4), the map u extends continuously to infinity. In particular, there is a point $x \in \Phi^{-1}(0)$ such that $\lim_{r \rightarrow \infty} u(r, \theta) = e^{\eta\theta} x$ for all $\theta \in [0, 2\pi]$. Choose a gauged pair (A', u') (not necessarily holomorphic) in the following way. The connection A' is in $W_{loc}^{1,p}$ and on $\mathbb{C} \setminus B_R$, $A' = d + \eta d\theta$. The map u' satisfies $u'(re^{i\theta}) = e^{\eta\theta} x$ for $r \geq R$ and is obtained from u by an arbitrary continuous homotopy, such that the image of the homotopy is contained in a compact subset of X . We denote the operator in (42) by $L_{(A,u)}$ and we define another analogous operator

$$\mathcal{L}_{(A',u')} : W^{k+1,p,\delta}(\mathbb{C}, \mathfrak{g}) \rightarrow W^{k-1,p,\delta}(\mathbb{C}, \mathfrak{g}), \quad \zeta \mapsto d_{A'}^* d_{A'} \zeta + d\Phi_u(J\zeta_u).$$

The difference is

$$(44) \quad \mathcal{L}_{(A,u)} - \mathcal{L}_{(A',u')} = (d_A^* d_A - d_{A'}^* d_{A'}) + (d\Phi_u(\cdot_u) - d\Phi_{u'}(\cdot_{u'})).$$

By Proposition 2.10, both terms in the rhs of (44) above are compact. We show details for the first term. Recall that for two connections A and $A+a$, the difference between the Hodge Laplacians of the covariant derivatives is

$$(d_{A+a}^* d_{A+a} - d_A^* d_A) \zeta = *[a \wedge *d_A \zeta] + d_A^*[a \wedge \zeta] + *[a \wedge *[a \wedge \zeta]],$$

where ζ is a section of the associated bundle $P(\mathfrak{g})$. Returning to our case, on $\mathbb{C} \setminus B_R$, $A = d + \eta d\theta + a$ and $A' = d + \eta d\theta$. So, we get

$$(45) \quad (d_A^* d_A - d_{A'}^* d_{A'}) \zeta = *[a \wedge *d_A \zeta] + d_A^*[a \wedge \zeta] + *[a \wedge *[a \wedge \zeta]].$$

Consider the second term, and expand it as

$$(46) \quad \begin{aligned} d_A^*[a \wedge \zeta] &= d^*[a \wedge \zeta] + *[\eta d\theta \wedge *[a \wedge \zeta]] \\ &= *[d(*a) \wedge \zeta] + *[*a \wedge d\zeta] + *[\eta d\theta \wedge *[a \wedge \zeta]]. \end{aligned}$$

By Proposition 2.10, $\zeta \mapsto d_A^*[a \wedge \zeta]$ is a compact operator between the spaces $W^{k+1,p,\delta}(\mathbb{C}, \mathfrak{g}) \rightarrow W^{k-1,p,\delta}(\mathbb{C}, \mathfrak{g})$.

The operator $\mathcal{L}_{(A',u')}$ is an isomorphism and the operators $\mathcal{L}_{(A,u)}$ and $\mathcal{L}_{(A',u')}$ differ by a compact perturbation. Therefore $\mathcal{L}_{(A,u)}$ is Fredholm and has index zero. So, it is enough to prove that $\mathcal{L}_{(A,u)}$ is injective in order to finish the proof of the Proposition. For any $\zeta \in W^{k+1,p,\delta}$, integrating by parts, we get

$$(47) \quad \int_{\mathbb{C}} \langle d_A^* d_A \zeta + d\Phi_u(J\zeta_u), \zeta \rangle = \|d_A \zeta\|_{L^2(\mathbb{C})}^2 + \int_{\mathbb{C}} \omega_u(\zeta_u, J_u \zeta_u) \, d\text{vol}_{\mathbb{C}}.$$

The rhs is non-negative and vanishes exactly if $d_A \zeta$ and ζ_u vanish. Since the G -action is free for points in $u(\mathbb{C} \setminus B_R)$, we can conclude that if $d_A \zeta$ and ζ_u vanish, then $\zeta = 0$. The only problem with this computation is that we are yet to verify if the above integrals converge and that the boundary term in the integration by parts vanishes. We verify this in the rest of the proof. Consider the lhs of (47). We restrict attention to the case $k = 0$, as the $k = 1$ case is similar. Since $\mathcal{L}_{(A,u)} \zeta \in W^{-1,p,\delta}$, the integral is finite if $\zeta \in W^{1,p',-\delta}$, where $p' = \frac{p}{p-1}$ is the conjugate exponent, and this result is a consequence of Lemma 7.6. In a similar way, the other terms can be shown to converge. The boundary term is given by $\int_{\partial B_r} \langle *d_{A,r} \zeta, \zeta \rangle$, where $d_{A,r} \zeta$ is the radial component of $d_A \zeta$. This term decays to zero as $r \rightarrow \infty$. \square

Lemma 7.6. *Suppose $p > 2$, $\delta > 1 - \frac{2}{p}$ and $p' = \frac{p}{p-1}$ is the conjugate exponent of p . Then, $L^{p,\delta}(\mathbb{C}) \hookrightarrow L^{p',-\delta}(\mathbb{C})$.*

Proof. Define q as $\frac{1}{q} = 1 - \frac{2}{p}$. Then, $\delta > 1 - \frac{2}{p}$ implies that $(1 + |z|^2)^{-\delta} \in L^q$. Then by Höder's inequality,

$$\begin{aligned} \|(1 + |z|^2)^{-\delta/2} f\|_{L^{p'}}^p &= \int_{\mathbb{C}} (1 + |z|^2)^{-p'\delta} (1 + |z|^2)^{p'\delta/2} |f|^{p'} \\ &\leq \|(1 + |z|^2)^{-\delta}\|_{L^q}^{p'/q} \|f\|_{L^{p,\delta}}^{p'/p}. \end{aligned}$$

□

The following version of the implicit function theorem is Proposition A.3.4 in [11], and is used in the proof of Proposition 7.1.

Proposition 7.7. (Implicit function theorem) *Let $F : X \rightarrow Y$ be a differentiable map between Banach spaces. Suppose that $DF(0)$ is surjective and has a right inverse Q , with $\|Q\| \leq C$. For all $x \in B_\delta$, $\|DF(x) - DF(0)\| < \frac{1}{2C}$. If $\|F(0)\| < \frac{\delta}{4C}$, then $F(x) = 0$ has a unique solution in B_δ satisfying $x \in \text{Im } Q$.*

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